ENUMERATION PROBLEMS FOR CLASSES OF SELF-SIMILAR GRAPHS

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Abstract. We describe a general construction principle for a class of self-similar graphs. For various enumeration problems, we show that this construction leads to polynomial systems of recurrences and provide methods to solve these recurrences asymptotically. This is shown for different examples involving classical self-similar graphs such as the Sierpiński graphs. The enumeration problems we investigate include counting independent subsets, matchings and connected subsets.

1. Introduction

Counting sets satisfying a fixed property in graphs ranges among the classical tasks of combinatorics. There is a vast amount of literature on this kind of combinatorial problems for various classes of graphs, especially for trees, by different authors. We note for instance the following set counting problems which have been studied in the past:

- the number of independent or maximal independent subsets [7, 8, 18, 16, 17, 26, 29],
- the number of subtrees of a tree [18, 25, 33, 37],
- the number of matchings or maximal matchings [6, 10, 11, 18],
- the number of chains/antichains in a tree [18, 25].

All these graph invariants reflect the structure of a graph in some way, and therefore, some of them are even of interest in theoretical chemistry for the study of molecular graphs (see [32, 38]). For example, the number of independent subsets is called Merrifield-Simmons-index, the number of matchings is known as Hosoya-index in chemistry. It was shown that both correlate well with physicochemical properties of the corresponding molecules (see [13, 27]).

A special type of self-similar graph that has been of interest is the complete t-ary tree [19]. It is constructed in the following way:

- Start with a single vertex (the root) to obtain the level-zero tree T_0,
- take t copies of T_n and connect their roots to a new common root to obtain T_{n+1}.

A natural reason to study complete t-ary trees is that they are usually extremal with respect to the cited graph invariants among all trees of bounded degree. The number of independent sets in these graphs has been investigated in [16], the number of subtrees in [37]. In this paper, the stated way of construction is formalized and generalized.

Other examples of graphs with self-similar properties – even though they are not among the class generated by our construction principle – that appear in applications are the rectangular and hexagonal grid graphs. For example, the growth of the number of independent sets in a $m \times n$-grid is of interest in statistical physics (see [5]). It is known that the number of independent sets in a (n, m)-grid graph grows with α^{mn}, where $\alpha = 1.503048082\ldots$ is the so-called hard square entropy constant. The bound for this constant was successively improved by Weber [39], Engel [8] and Calkin and Wilf [7].

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The calculation of asymptotic formulas of this type is the main aim of this paper – usually, in our examples, we will observe a doubly exponential growth, where the implied constants can only be calculated numerically. Recursively defined sequences with doubly exponential growth have been investigated, for instance, by Aho and Sloane [1] and Ioanescu and Stanica [14]. The formula for the sequence defined by $x_0 = 1$, $x_n = (x_{n-1} + 1)^2$ given in [1] ($x_n = \lfloor \alpha^{2^n} \rfloor - 1$, where $\alpha = 2.258518\ldots$) has been used by Székely and Wang to determine the number of subtrees in a complete binary tree [37].

In our final example connected subsets in finite Sierpiński graphs are counted. Besides the usual doubly exponential growth, an unusual exponential factor appears in the asymptotic formula. The base of this exponential factor is apparently the same as the resistance scaling factor of the infinite Sierpiński graph, see [3] for the definition of this constant. This indicates connections between the number of connected subsets in finite self-similar graphs and energy forms, Laplacians and random walks on the associated infinite graphs.

Fractal spaces, especially the Sierpiński gasket, were first considered as interesting state spaces in stochastics in physical literature, see for example [2, 30, 31]. This work was continued by the rigorous development of Brownian motion on self-similar sets, see [3] and the references therein. In all this research the approximation of fractal sets by fractal-like graphs is of vital importance. Therefore graphs obeying some fractal law were studied in many respects: See for example [12, 15, 40] and the references therein for publications on spectra of fractal-like graphs, analysis and stochastics.

The notion of graphical substitution is the basic construction principle for fractal-like graphs; however, there is no unified theory; see [20, 24, 34]. In the following we define a very general construction scheme, which can be applied to many classical examples, including self-similar graphs and trees with finitely many cone-types, see [28, 34]. The self-similar nature of graphs in this class is then the starting point for investigations on the aforementioned counting problems. It turns out that the substitution procedure translates to dynamical systems for the combinatorial quantities.

2. Construction

We recall some basic definitions about graphs: Graphs $X = (VX, EX)$ with vertex set VX and edge set EX are always supposed to be undirected, without loops or multiple edges. Vertices x and y are *adjacent* if $\{x, y\}$ is an edge in EX. The *degree* $\deg_X(x)$ of x is the cardinality of vertices in VX being adjacent to x.

Let X be a graph and $\sim$ be an equivalence relation on VX. We write $VX/{\sim}$ for the set of all equivalence classes of $\sim$ and denote by $\overline{v}$ the equivalence class of the vertex $v \in VX$. Then the graph $Y = X/{\sim}$ is defined by $VY = VX/{\sim}$ and

$$EY = \{\{\overline{v}, \overline{w}\} \,:\, v, w \in VX,\ \{v, w\} \in EX\}.$$

In the following we describe a substitutional graph construction, which resembles the construction of graph-directed self-similar sets: Fix a number $m \in \mathbb{N}$ and let the following data be given:

- Initial graphs $X_1, \ldots, X_m$.
- Distinguished vertices on each initial graph. For $k \in \{1, \ldots, m\}$ the distinction is given as a map $\phi_k : \{1, 2, \ldots, \theta(k)\} \to VX_k$, where $\theta(k) \geq 1$ is the number of distinguished vertices in X_k.
- Model graphs $G_1, \ldots, G_m$.
- A map $\psi_k : \{1, 2, \ldots, \theta(k)\} \to VG_k$, which defines $\theta(k)$ distinguished vertices on G_k.
- The number $s(k) \geq 1$ of substitutions associated to the model graph G_k for $k \in \{1, \ldots, m\}$ and a map $\tau_k : \{1, \ldots, s(k)\} \to \{1, \ldots, m\}$, which describes the type of substitution. Last but not least, one-to-one maps $\sigma_{k,i} : \{1, \ldots, \theta(\tau_k(i))\} \to VG_k$ for $k \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, s(k)\}$, which describe each substitution.

With this data we inductively construct m sequences $(X_{k,n})_{n\geq 0}$ of graphs and maps $\phi_{k,n} : \{1,\ldots,\theta(k)\} \to VX_{k,n}$, which define distinguished vertices of the graph $X_{k,n}$: For $k \in \{1,\ldots,m\}$ and $n = 0$ set $X_{k,0} = X_k$ and $\phi_{k,0} = \phi_k$. Now fix $n > 0$ and $k \in \{1,\ldots,m\}$. For $i \in \{1,\ldots,s(k)\}$ let $Z_{k,n,i}$ be an isomorphic copy of the graph $X_{\tau_k(i),n-1}$, where the isomorphism is given by $\gamma_{k,n,i} : X_{\tau_k(i),n-1} \to Z_{k,n,i}$. Additionally, we require that the vertex sets VG_k and $VZ_{k,n,1},\ldots,VZ_{k,n,s(k)}$ are mutually disjoint. Now let $Y_{k,n}$ be the disjoint union of the graphs G_k and $Z_{k,n,1},\ldots,Z_{k,n,s(k)}$ and define the relation $\sim$ on the vertex set $VY_{k,n}$ to be the reflexive, symmetric and transitive hull of

$$\bigcup_{i=0}^{s(k)} \Big\{ \{\sigma_{k,i}(j), \gamma_{n,k,i}(\phi_{k,n-1}(j))\} \,:\, j \in \{1,\ldots,\theta(\tau_k(i))\} \Big\} \subseteq VY_{k,n} \times VY_{k,n}.$$

Then $X_{k,n} = Y_{k,n}/{\sim}$ and the map $\phi_{k,n}$ is defined by $\phi_{k,n}(i) = \overline{\psi_k(i)} \in VX_{k,n}$. Furthermore, we call the subgraph $\overline{Z_{k,n,i}}$ of $X_{k,n}$ (which is isomorphic to $X_{\tau_k(i),n-1}$) the *i-th part* of $X_{k,n}$.

Remark. Note that edges of two distinct graphs $Z_{k,n,i}$ and $Z_{k,n,j}$ may be amalgamated in $X_{k,n}$. In the rest of this paper we require that this is not case. For example, this can be achieved if

$$|\sigma_{k,i}(\{1,\ldots,\theta(\tau_k(i))\}) \cap \sigma_{k,j}(\{1,\ldots,\theta(\tau_k(j))\})| \leq 1$$

holds for any k and distinct $i,j \in \{1,\ldots,s(k)\}$. This means, that two distinct parts of $X_{k,n}$ never have more than one vertex in common.

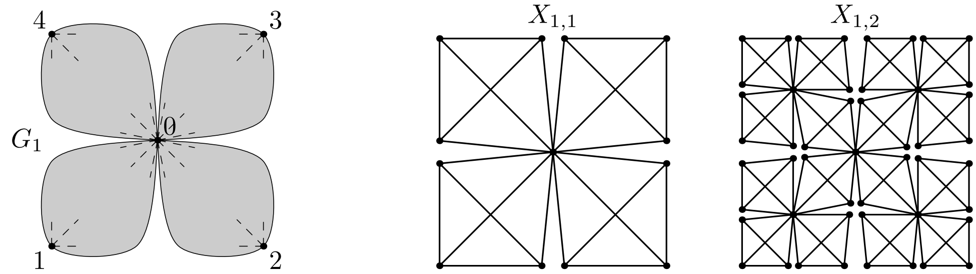

FIGURE 1. Model graph and $X_{1,1}$, $X_{1,2}$.

Example 1. Fix some integers $p,q \in \mathbb{N}$. Let $m = 1$, $\theta(1) = 2$ and $X_1 = K_p$. Let $x,y \in VX_1$ be two different vertices and set $\phi_1(1) = x$ and $\phi_1(2) = y$. Let G_1 be given by $VG_1 = \{0,\ldots,q\}$ and $EG_1 = \emptyset$, and define $\psi_1(1) = 1$ and $\psi_1(2) = 2$. Finally, let $s(1) = q$ and $\sigma_{1,i}(1) = 0$, $\sigma_{1,i}(2) = i$ for $i \in \{1,\ldots,q\}$. See Figure 1 for the case $p = q = 4$. The associated infinite graphs were studied in [20, 21, 22] concerning growth, spectral properties and behavior of random walk.

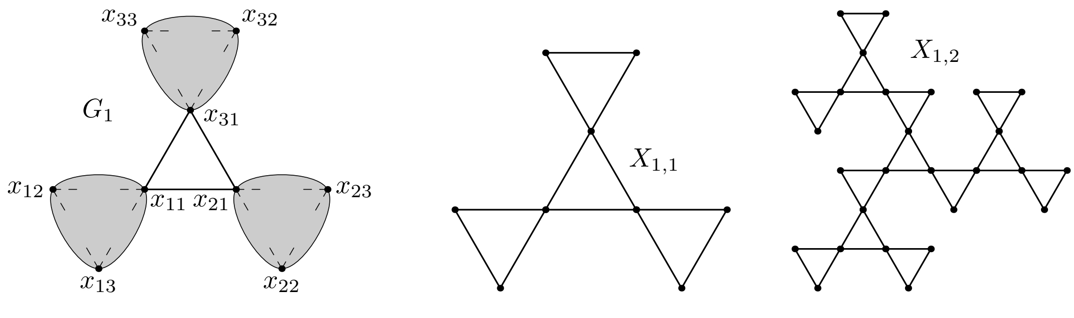

FIGURE 2. Model graph and $X_{1,1}$, $X_{1,2}$.

Example 2 (The loop-erased Schreier graph of the Fabrykowski-Gupta group). Let $m = 1$ and let $X_1 = K_3$, where $VX_1 = \{1,2,3\}$. Let $\theta(1) = 3$ and $\phi_1(i) = i$ for $i \in \{1,2,3\}$. Furthermore, define G_1 by

$$VG_1 = \{x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}\},$$
$$EG_1 = \big\{\{x_{11},x_{21}\},\{x_{21},x_{31}\},\{x_{31},x_{11}\}\big\},$$

and set $\psi_1(i) = x_{i2}$ for $i \in \{1, 2, 3\}$. Finally we set $s(1) = 3$ and $\sigma_{1,i}(j) = x_{ij}$. See Figure 2 for a visualization of the model graph G_1 and $X_{1,1}$, $X_{1,2}$. The Fabrykowski-Gupta group was introduced in [9], the corresponding Schreier graph was studied in [4]; see also [12].

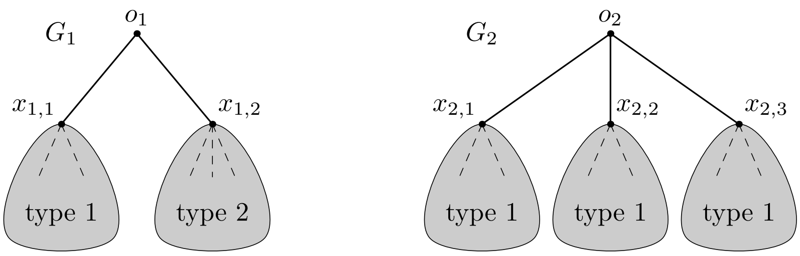

FIGURE 3. The model graphs G_1 and G_2.

Example 3 (Trees with finitely many cone types). Let $A = (a_{ij})$ be an $m \times m$ matrix with non-negative integer entries. For $k \in \{1, \ldots, m\}$ let $X_k = (\{x\}, \emptyset)$, $\theta(k) = 1$, and $\phi_k(1) = x$. Furthermore, let G_k be a star with root $o_k \in VG_k$ and $s(k) = a_{k1} + \cdots + a_{km}$ leaves $\{x_{k,1}, \ldots, x_{k,s(k)}\}$, and let $\psi_k(1) = o_k$. Finally, we set $\tau_k(j) = t$ if

$$\sum_{i=1}^{t-1} a_{ki} < j \leq \sum_{i=1}^{t} a_{ki}$$

and $\sigma_{k,i}(1) = x_{k,i}$. Then the graphs $X_{k,n}$ constructed as above describe finite analoga of infinite trees with finitely many cone types, see [28] and the references therein. Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 3 & 0 \end{smallmatrix}\right)$, then Figure 3 shows a visualization of the substitution procedure.

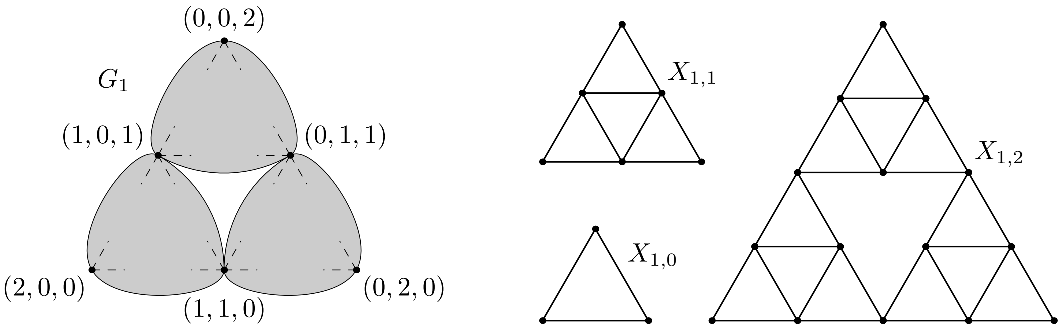

FIGURE 4. Model graph and finite Sierpiński graphs.

Example 4 (Sierpiński graphs, see [35]). Let $m = 1$ and fix some $d \in \mathbb{N}_0$. We define X_1 and G_1 by

$$VX_1 = \Big\{ \mathbf{x} = (x_0, \ldots, x_d) \in \mathbb{N}_0^{d+1} : \sum_{i=0}^{d} x_i = 1 \Big\}, \qquad EX_1 = \{\{\mathbf{x}, \mathbf{y}\} : \|\mathbf{x} - \mathbf{y}\|_1 = 2\}$$

and

$$VG_1 = \Big\{ \mathbf{x} = (x_0, \ldots, x_d) \in \mathbb{N}_0^{d+1} : \sum_{i=0}^{d} x_i = 2 \Big\}, \qquad EG_1 = \emptyset,$$

respectively. Let $\theta(1) = d + 1$ and $\phi_1(i) = \mathbf{e}_i$, $\psi_1(i) = 2\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical basis vector. In addition, let $s(1) = d$ and $\sigma_{1,i}(j) = \mathbf{e}_i + \mathbf{e}_j$. See Figure 4 for the case $d = 3$.

Example 5. Let $m = 1$ and let X_1 be any finite connected graph with at least two vertices x_1, x_2. We set $\theta(1) = 2$ and define $\phi_1(1) = x_1$ and $\phi_2(1) = x_2$. Let G_1 be any finite edgeless graph with at least two vertices v_1, v_2 and define $\psi_1(1) = v_1$ and $\psi_1(2) = v_2$. Furthermore, choose $s(1) = s$ mutually distinct pairs of vertices $(w_{1,1}, w_{1,2}), \ldots, (w_{s,1}, w_{s,2})$ in $VG_1 \times VG_1$, so that $w_{j,1} \neq w_{j,2}$ for $j \in \{1, \ldots, s\}$, and define $\sigma_{1,1}(j) = w_{j,1}$ and $\sigma_{1,2}(j) = w_{j,2}$ for $j \in \{1, \ldots, s\}$. We note that connectedness of the graphs $X_{1,1}, X_{1,2}, \ldots$ can be guaranteed if the pairs $(w_{1,1}, w_{1,2}), \ldots, (w_{s,1}, w_{s,2})$ induce a connected directed graph structure on G_1. See Figure 5 for a

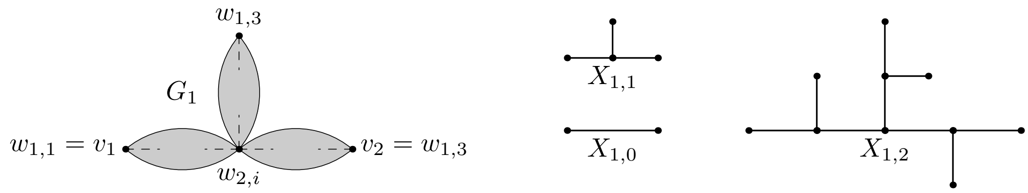

FIGURE 5. Model graph and $X_{1,0}$, $X_{1,1}$, $X_{1,2}$.

simple example in this class. Spectral properties of the associated infinite graphs were investigated in [24].

3. Types of Enumeration Problems

Our aim will be to solve enumeration problems in graphs of the type we defined in the previous section. We want to count the number of certain combinatorial objects (typically, sets of vertices or edges) satisfying a given property, such as independency or connectivity. Our method of solving these problems works for all properties satisfying some compatibility axioms which are presented in this section; these axioms guarantee us that we can establish recurrence equations reflecting the recursive construction of our graphs.

Let $\mathcal{C}(X)$ denote a family of combinatorial objects associated to a graph X. We want to count the number of elements $c \in \mathcal{C}(X_{k,n})$ (with the notation of the previous section) satisfying a certain property P. The set of all these elements is denoted by $\mathcal{C}(X_{k,n} \,|\, P)$. We suppose that for each $k \in \{1, \dots, m\}$ there are finitely many properties $P_{k,r}$, $r \in \{1, \dots, R_k\}$, of elements in $\mathcal{C}(X_{k,n})$ and subsets

$$B_{k,r} \subseteq \mathcal{C}(G_k) \times \prod_{i=1}^{s(k)} \{1, \dots, R_{\tau_k(i)}\},$$

so that P can be expressed in terms of $P_{k,r}$ and there exists a bijective correspondence between

$$\mathcal{C}(X_{k,n} \,|\, P_{k,r}) \qquad \text{and} \qquad \biguplus_{(b,r_1,\dots,r_{s(k)}) \in B_{k,r}} \{b\} \times \prod_{i=1}^{s(k)} \mathcal{C}(X_{\tau_k(i),n-1} \,|\, P_{\tau_k(i),r_i}). \tag{1}$$

Less formally spoken, the property $P_{k,r}$ can be reduced to properties on the parts of $X_{k,n}$; given $(b, r_1, \dots, r_{s(k)}) \in B_{k,r}$ and objects $c_1, \dots, c_{s(k)}$ belonging to the parts of $X_{k,n}$, so that $c_i \in \mathcal{C}(X_{\tau_k(i),n-1} \,|\, P_{\tau_k(i),r_i})$, one can construct a unique object c with property $P_{k,r}$ from them, and the correspondence is bijective. Note that the same $s(k)$-tuple $(c_1, \dots, c_{s(k)})$ may appear more than once.

Example 6. We give a short example for illustration: let $\mathcal{C}(X_{k,n})$ be the family of vertex subsets of $X_{k,n}$, and let $P(c)$ be the property that the set c is an independent set, i.e. there is no pair of adjacent vertices in c. For the sake of notation set $\Theta(k) = \{1, \dots, \theta(k)\}$. Then we may define our properties in the following way: for all k and all subsets S of $\Theta(k)$, let a set $c \in \mathcal{C}(X_{k,n})$ with property $P_{k,S}$ be an independent set such that

$$c \cap \phi_{k,n}(\Theta(k)) = \phi_{k,n}(S).$$

Thus c contains exactly the distinguished vertices corresponding to elements of S. Clearly, P is the union of all of these properties. For $k \in \{1, \dots, m\}$, $i \in \{1, \dots, s(k)\}$ and $S \subseteq \Theta(k)$ set

$$\rho_{k,i}(S) = \{j \in \Theta(\tau_k(i)) : \sigma_{k,i}(j) \in \psi_k(S)\}.$$

So $\rho_{k,i}(S)$ corresponds to distinguished vertices of the i-th part of $X_{k,n}$, which are also distinguished vertices in $X_{k,n}$ itself. Then we define $B_{k,S}$ by

$$B_{k,S} = \mathcal{C}(G_k \,|\, Q_{k,S}) \times \prod_{i=1}^{s(k)} \{T \subseteq \Theta(\tau_k(i)) : T \cap \rho_{k,i}(\Theta(k)) = \rho_{k,i}(S)\}.$$

Here, a set b with property $Q_{k,S}$ in G_k is an independent subset such that

$$b \cap \psi_{k,n}(\Theta(k)) = \psi_{k,n}(S).$$

An independent set c in $X_{k,n}$ with property $P_{k,S}$ induces an independent set $b \in \mathcal{C}(G_k)$ and independent sets $c_1, \dots, c_{s(k)}$ in all parts of $X_{k,n}$. By the choice of S, it is fixed for the distinguished vertices of the i-th part whether they belong to c_i or not, so the c_i must satisfy properties of the form $P_{\tau_k(i),R}$. Conversely, given $b \in \mathcal{C}(G_k \,|\, Q_{k,S})$ and independent subsets in all parts of $X_{k,n}$ (with appropriately fixed distinguished vertices), one can construct a unique independent set with property $P_{k,S}$ from them.

A more intuitive description will be given in the examples of Section 5. The interested reader may check that each of the following properties can be handled in a similar way and thus meets with our requirements:

- matchings (independent edge subsets),
- connected subsets,
- subtrees or spanning subtrees,
- colorings,
- factors,
- vertex or edge coverings,
- maximal independent sets,
- maximal matchings.

The latter two need some additional care, but the reduction process works for them, too.

4. Polynomial recurrence equations

The benefit we take from the axioms of the preceding chapter is simple: it is easy now to derive recursive relations for the cardinalities of the sets $\mathcal{C}(X_{k,n} \,|\, P_{k,r})$. Let $c_n(k,r) := |\mathcal{C}(X_{k,n} \,|\, P_{k,r})|$. From the bijective correspondence (1) we immediately conclude that

$$c_n(k,r) = \sum_{(b,r_1,\dots,r_{s(k)}) \in B_{k,r}} \prod_{i=1}^{s(k)} c_{n-1}(\tau_k(i), r_i)$$

for $k \in \{1, \dots, m\}$ and $r \in \{1, \dots, R_k\}$. Now, all $c_n(k,r)$ can be obtained from the initial values $c_0(k,r)$ and this system of polynomial recurrence equations. In the following, we will show how to obtain asymptotic properties of the sequences $c_n(k,r)$ from such a system.

Proposition 1. *Let $\mathbf{p} : \mathbb{R}^d \to \mathbb{R}^d$ be a non-linear polynomial function with non-negative coefficients and $\mathbf{c}_0 \in \mathbb{R}^d$, so that $c_{0,i} > 0$ for all $i \in \{1, \dots, d\}$. Define the orbit sequence $(\mathbf{c}_n)_{n \geq 0}$ by $\mathbf{c}_{n+1} = \mathbf{p}(\mathbf{c}_n)$ for $n \in \mathbb{N}_0$. We assume that $c_{n,i}$ tends to ∞ as $n \to \infty$ for all $i \in \{1, \dots, d\}$ and $c_{n,i} \asymp c_{n,j}$ as $n \to \infty$ holds for all $i, j \in \{1, \dots, d\}$. Then $c_{n,i} = \exp(Kq^n + O(1))$ for all $i \in \{1, \dots, d\}$, where $q > 1$ is the total degree of $\mathbf{p}$ and $K > 0$ is some constant.*

Proof. Let $\mathbf{p} = (p_1, \dots, p_d)$ and choose $k \in \{1, \dots, d\}$, so that the total degree q of p_k is strictly larger than 1. By the conditions of the sequence $(\mathbf{c}_n)_{n \geq 0}$ there are $\mathbf{r}_n \in \mathbb{R}^d$, so that $c_{n,i} = r_{n,i} c_{n,k}$, and the set $\{r_{n,i} : n \in \mathbb{N}_0, i \in \{1, \dots, d\}\}$ is bounded from below and above by positive constants. In the following we use multi-index notation: let

$$p_k(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}.$$

This implies

$$c_{n+1,k} = p_k(\mathbf{c}_n) = \sum_{j=0}^{q} b_{n,j} c_{n,k}^j,$$

where the coefficients $b_{n,j}$ are defined by

$$b_{n,j} = \sum_{|\mathbf{i}|=j} a_{\mathbf{i}} \mathbf{r}_n^{\mathbf{i}}.$$

Notice that $\{b_{n,q} : n \in \mathbb{N}_0\}$ is bounded from below and above by positive constants and that $\{b_{n,j} : n \in \mathbb{N}_0\}$ is bounded above for all $j < q$. Write $x_n = \log(c_{n,k})$, then

$$x_{n+1} = qx_n + d_n, \tag{2}$$

where d_n is given by

$$d_n = \log\left(\sum_{j=0}^{q} b_{n,j} c_{n,k}^{j-q}\right).$$

Since $c_{n,k}$ tends to ∞, the numbers d_n are bounded. Now Equation (2) implies

$$x_n = q^n\left(x_0 + \sum_{\ell=0}^{n-1} \frac{d_\ell}{q^{\ell+1}}\right) = q^n\left(x_0 + \sum_{\ell=0}^{\infty} \frac{d_\ell}{q^{\ell+1}} + O(q^{-n})\right).$$

Define K by

$$K = x_0 + \sum_{\ell=0}^{\infty} \frac{d_\ell}{q^{\ell+1}},$$

then $c_{n,1} = \exp(Kq^n + O(1))$ follows. This implies the statement. Furthermore, we remark that the total degree of p_i must be q for any $i \in \{1, \ldots, d\}$. ∎

Remark. With the notation of the previous proof we notice that the asymptotic behavior of the sequence $(\mathbf{c}_n)_n$ is mostly determined by those monomials of $\mathbf{p}$ of total degree q. By the previous result there are vectors $\mathbf{C}_n \in \mathbb{R}^d$ (bounded above and below) such that $\mathbf{c}_n = \mathbf{C}_n \exp(Kq^n)$. Now write $\mathbf{p} = \mathbf{h} + \mathbf{r}$, where all monomials of $\mathbf{h}$ have total degree q and the total degree of $\mathbf{r}$ is strictly smaller than q. So h is a homogeneous polynomial of degree q. Then

$$\begin{aligned}
\mathbf{C}_{n+1} \exp(Kq^{n+1}) &= \mathbf{c}_{n+1} = \mathbf{p}(\mathbf{c}_n) = \mathbf{h}(\mathbf{c}_n) + \mathbf{r}(\mathbf{c}_n) \\
&= \mathbf{h}(\mathbf{C}_n \exp(Kq^n)) + \mathbf{r}(\mathbf{C}_n \exp(Kq^n)) \\
&= \exp(Kq^{n+1})\mathbf{h}(\mathbf{C}_n) + O(\exp(K(q-1)q^n)).
\end{aligned}$$

This implies $\mathbf{C}_{n+1} = \mathbf{h}(\mathbf{C}_n) + O(\exp(-Kq^n))$. In order to obtain information about $\mathbf{C}_n$ we have to study the dynamical system associated to the map $\mathbf{h}$. The case $\mathbf{r} \equiv \mathbf{0}$ is of special interest: on the one hand it occurs in the given examples, on the other hand the error term disappears leading to $\mathbf{C}_{n+1} = \mathbf{h}(\mathbf{C}_n)$. Thus, in this case we have to investigate the dynamical behavior of $\mathbf{h}$ in the projective space.

Proposition 2. *Let $\mathbf{p} : \mathbb{R}^d \to \mathbb{R}^d$ be a homogeneous polynomial of degree $q > 1$ with an attracting fixed point $\mathbf{C} \neq \mathbf{0}$. Let $\mathbf{c}_0 \in \mathbb{R}^d$ and define $\mathbf{c}_{n+1} = \mathbf{p}(\mathbf{c}_n)$ for $n \in \mathbb{N}_0$. We assume that $(\mathbf{c}_n)_{n\geq 0}$ defines a sequence in the projective space $\mathbb{P}^{d-1}$ converging to $\mathbf{C}$ in $\mathbb{P}^{d-1}$. Then $\mathbf{c}_n = \mathbf{C}\exp(Kq^n + o(1))$ for some $K \in \mathbb{R}$.*

Proof. As $\mathbf{c}_n \to \mathbf{C}$ in $\mathbb{P}^{d-1}$ there are $r_n \neq 0$ such that $r_n\mathbf{c}_n \to \mathbf{C}$ in $\mathbb{R}^d$. Thus the sequence $\boldsymbol{\varepsilon}_n = r_n\mathbf{c}_n - \mathbf{C}$ converges to $\mathbf{0}$. Define $\mathbf{u}_n$ by $\mathbf{u}_n = \mathbf{p}(\mathbf{C} + \boldsymbol{\varepsilon}_n) - \mathbf{C}$. As $\mathbf{C}$ is an attracting fixed point of $\mathbf{p}$, the sequence $(\mathbf{u}_n)_{n\geq 0}$ converges to $\mathbf{0}$, too. An easy computation yields

$$r_n^q(\mathbf{C} + \boldsymbol{\varepsilon}_{n+1}) = r_{n+1}(\mathbf{C} + \mathbf{u}_n).$$

There exists a $k \in \{1, \ldots, d\}$ with $C_k \neq 0$. Choose n_0 sufficiently large, so that

$$d_n = \frac{C_k + \varepsilon_{n+1,k}}{C_k + u_{n,k}}$$

satisfies $|d_n - 1| < \frac{1}{2}$ for all $n \geq n_0$. Notice that $d_n \to 1$. This implies

$$\log(r_n) = q\log(r_{n-1}) + \log(d_{n-1}) = q^n\left(q^{-n_0}\log(r_{n_0}) + \sum_{\ell=n_0}^{n-1} \frac{\log(d_\ell)}{q^{\ell+1}}\right) = -Kq^n + o(1),$$

where K is given by

$$K = -q^{-n_0} \log(r_{n_0}) - \sum_{\ell=n_0}^{\infty} \frac{\log(d_\ell)}{q^{\ell+1}}.$$

Therefore $r_n = \exp(-Kq^n + o(1))$. Since $r_n \mathbf{c}_n \to \mathbf{C}$, we obtain $\mathbf{c}_n = \mathbf{C} \exp(Kq^n + o(1))$. ∎

Remark. The last proposition can be generalized to the case of an attracting cycle $\mathbf{C}_1, \dots, \mathbf{C}_m$ of $\mathbf{p}$. If the sequence $(\mathbf{c}_n)$ is attracted by this cycle in $\mathbb{P}^{d-1}$, then an adapted version of the result above holds.

The preceding propositions give us the necessary tools to cope with a variety of set-counting problems for classes of self-similar graphs. Unfortunately, they are not applicable to all conceivable problems of that kind. In can be seen especially from the example of section 5.3 that there is a vast variety of possibilities for the asymptotical behavior of a polynomial recurrence system.

5. Examples

5.1. **Matchings, maximal matchings and maximum matchings.** We turn to Example 2 of Section 2 now. The sequence of graphs that was constructed in this example has some particularly nice properties in connection with matchings, therefore, we present the problem of enumerating the matchings on this graph here. First, we consider ordinary matchings.

Let $m_{0,n}$ be the total number of matchings in the level-n graph $X_{1,n}$ of the construction described in Example 2 of Section 2. Furthermore, let $m_{1,n}$ be the number of matchings with the property that a fixed vertex from the set of distinguished vertices is unmatched, and let $m_{2,n}$ be the number of matchings with the property that two fixed vertices from the set of distinguished vertices are unmatched. By symmetry, it is not relevant which of the distinguished vertices we choose.

It is easy to see that $m_{0,0} = 4$, $m_{1,0} = 2$ and $m_{2,0} = 1$, and that the following system of recurrence equations holds (we only have to consider four cases for the center triangle – either none of the edges of the center triangle belongs to the matchings or one of the three belongs to it):

$$\begin{aligned} m_{0,n+1} &= m_{0,n}^3 + 3m_{0,n}m_{1,n}^2, \\ m_{1,n+1} &= m_{0,n}^2 m_{1,n} + m_{1,n}^3 + 2m_{0,n}m_{1,n}m_{2,n}, \\ m_{2,n+1} &= m_{0,n}m_{1,n}^2 + m_{0,n}m_{2,n}^2 + 2m_{1,n}^2 m_{2,n}, \end{aligned}$$

A straightforward induction shows us that $m_{0,n} = 2m_{1,n} = 4m_{2,n}$ holds for all n. This can also be seen by an easy combinatorial argument:

Let v be one of the distinguished vertices (or any of the outermost vertices in the graph $X_{1,n}$), let v' be its neighbor of degree two, and let w be its neighbor of degree four. Clearly, the number of matchings containing the edge vv' is the same as the number of matchings in which v and v' are not matched at all. By symmetry, the number of matchings which match v is the same as the number of matchings which match v'. Altogether, this shows that the number of matchings which match v is exactly half of the total number of matchings.

The fact that the number of matchings which contain edges incident to two fixed distinguished vertices is exactly $\frac{1}{4}$ of the total number of matchings reflects the fact that the distinguished vertices (and, generally, arbitrary pairs of non-adjacent vertices which belong to the same orbit as the distinguished vertices) are independent with respect to the number of matchings – whether one of the vertices is to be matched or not does not affect the fraction of matchings in which the other is matched. This is due to the described bijections between matchings containing the edge vv' and those matching neither v nor v' respectively matchings containing vw and those containing wv'.

Thus, we only have to consider the simple recurrence equation $m_{0,n+1} = \frac{7}{4}m_{0,n}^3$ and $m_{0,0} = 4$, whose solution is given by

$$m_{0,n} = \frac{2}{\sqrt{7}}(2\sqrt{7})^{3^n}.$$

The first values of this sequence are 4, 112, 2458624, 26008445689991790592. So if $E = \frac{1}{2}(3^{n+2}-3)$ is the number of edges,

$$\left(\tfrac{16}{7}\right)^{1/3} 28^{E/9}$$

of the 2^E edge subsets are independent. The constant $28^{1/9}$ is approximately 1.4480892743...

Now, let us consider maximal matchings, i.e. matchings which cannot be extended any more. A little more care is needed for them, and some more variables as well. Again, we only have to consider two distinguished vertices and four cases for the edges in the middle triangle, but we have to consider three types of matchings with respect to a distinguished vertex $v = \phi_{1,n}(i)$:

- maximal matchings which match v,
- maximal matchings which leave v unmatched,
- matchings (not necessarily maximal) which leave v unmatched, with the additional property that every edge that can be added to the matching is incident to v.

Let us mark these properties by the numbers 0,1 and 2 respectively, and define sequences $M_{00,n}$, $M_{01,n}, \dots, M_{22,n}$, where, for instance, $M_{02,n}$ denotes the number of matchings in $X_{1,n}$ with the property that they contain an edge incident to one fixed distinguished vertex $v = \phi_{1,n}(i)$ and leave another fixed distinguished vertex $w = \phi_{1,n}(j)$ unmatched and can at most be extended by an edge containing w. By thoroughly distinguishing cases for the edges of the middle triangle, we obtain the general recurrence equation

$$\begin{aligned} M_{ij,n+1} = M_{i0,n}M_{j0,n}(M_{00,n}+M_{01,n}) + M_{i2,n}(M_{j0,n}+M_{j1,n})(M_{02,n}+M_{12,n}) \\ + (M_{i0,n}+M_{i1,n})M_{j2,n}(M_{02,n}+M_{12,n}) + M_{i1,n}M_{j0,n}(M_{00,n}+M_{01,n}) \\ + M_{i0,n}M_{j1,n}(M_{00,n}+M_{01,n}) + M_{i0,n}M_{j0,n}(M_{01,n}+M_{11,n}) \\ + M_{i2,n}M_{j2,n}(M_{00,n}+2M_{01,n}+M_{11,n}). \end{aligned}$$

together with the observation that, clearly, $M_{ij,n} = M_{ji,n}$. The initial values are given by $M_{00,0} = M_{01,0} = M_{02,0} = M_{22,0} = 1$ and $M_{11,0} = M_{12,0} = 0$. The total number of maximal matchings is given by $(M_{00,n} + 2M_{01,n} + M_{11,n})$. Now, if we regard the recursion for the $M_{ij,n}$ as a map in the projective space $\mathbb{P}^5$ of dimension 5, it is easy to check that every point of the algebraic surface

$$\big\{(x_{00},x_{01},x_{02},x_{11},x_{12},x_{22}) = (a^2,ab,ac,b^2,bc,c^2)\big\} \tag{3}$$

is a super-attractive fixed point of the dynamical system which is applied to the $M_{ij,n}$. In our case, the $M_{ij,n}$ tend (in projective space) to the following vector, which can be computed numerically:

$$(x_{00},x_{01},x_{02},x_{11},x_{12},x_{22}) = (0.390764, 0.162426, 0.292467, 0.0675145, 0.121568, 0.218897).$$

These values are chosen in such a way that the vector is also a fixed point of the system in $\mathbb{R}^6$. Now, by the observations of Section 4, in particular Proposition 2, we know that

$$\begin{aligned} &(M_{00,n}, M_{01,n}, M_{02,n}, M_{11,n}, M_{12,n}, M_{22,n}) \\ &\qquad \sim (0.390764, 0.162426, 0.292467, 0.0675145, 0.121568, 0.218897)\cdot \beta^{3^n} \end{aligned}$$

for some constant β, whose numerical value is $\beta = 3.3200219636\ldots$ (we skip the calculational details). So, the total number of maximal matchings in the graph $X_{1,n}$ we are considering is asymptotically $1.1682830147 \cdot (1.3055968738)^E$, where E denotes the number of edges again. The first values are 3, 29, 38375, 92180751403625, ... The parameterization (3) shows us that

$$\frac{M_{01,n}}{M_{00,n}} \sim \frac{M_{11,n}}{M_{01,n}} \sim \frac{M_{12,n}}{M_{02,n}}$$

and

$$\frac{M_{02,n}}{M_{00,n}} \sim \frac{M_{12,n}}{M_{01,n}} \sim \frac{M_{22,n}}{M_{02,n}},$$

so pairs of distinguished vertices are at least "asymptotically independent" of each other in this case. Roughly speaking, as the distance grows, the vertices do not interfere any more.

Finally, we observe that the graphs we studied within this section cannot have perfect matchings, since the number of vertices of $X_{1,n}$ is 3^{n+1}, an odd number. However, the following remarkable fact holds:

Theorem 3. *For every vertex v in the level-n graph $X_{1,n}$ of our construction, there is exactly one perfect matching in the graph $X_{1,n} \setminus v$.*

Proof. By induction on n. For $n = 0$, the theorem is essentially trivial. For the induction step, let, for the sake of brevity, P_1, P_2 and P_3 denote the parts which are joined by the center triangle, and let w_1, w_2, w_3 be the corresponding vertices of the center triangle. Without loss of generality, suppose that v belongs to P_3. Since P_1 contains an odd number of vertices, not all of the vertices of P_1 can be matched within P_1. The only vertex of P_1 which has neighbors outside P_1 is w_1, so w_1 is matched to either w_2 or w_3. Since the same holds true for w_2, w_1 and w_2 must be matched to each other. Now, the graph decomposes into the three parts, each reduced by exactly one vertex. By the induction hypothesis, we are done. ∎

Corollary 4. *The level-n graph $X_{1,n}$ of our construction has exactly 3^{n+1} maximum matchings (i.e. matchings of largest possible size), which equals the number of vertices.*

5.2. **Independent subsets in tree-like graphs.**

Theorem 5. *Let $p, q \geq 2$ be fixed integers, and define $X_{1,n}$ as in Example 1 of Section 2. Denote by a_n the number of independent vertex subsets of $X_{1,n}$. Then we have*

$$a_n \sim c_{p,q}\alpha_{p,q}^{q^n}$$

for some constants $\alpha_{p,q}$ and $c_{p,q}$. $\alpha_{p,q}$ can be estimated in the following way:

$$\left(2p^{q^2-q} + 2(p^q - p^{q-1} + 1)^q\right)^{q^{-2}} \leq \alpha_{p,q} \leq \left(p^{q^2-q} + (p^q - p^{q-1} + 1)^q\right)^{q^{-2}}.$$

Furthermore, $\alpha_{p,2} = \frac{1}{2}(-1 + p + \sqrt{5 - 2p + p^2})$ and $\alpha_{p,q}$ has a Laurent expansion around $p = \infty$, whose first terms are

$$\alpha_{p,q} = p - \frac{1}{q-1} + \frac{2-q}{2(q-1)^2} \cdot p^{-1} + \cdots$$

for $q > 2$.

Proof. We distinguish three cases for the number of independent vertex subsets, depending on the vertices $\phi_{1,n}(i)$ $(i = 1, 2)$:

- the number of independent vertex subsets containing none of these two vertices,
- the number of independent vertex subsets containing only $\phi_{1,n}(1)$ (by symmetry, this is the same as the number of independent vertex subsets containing only $\phi_{1,n}(2)$),
- the number of independent vertex subsets containing both of them.

We denote the first number by $a_{0,n}$, the second by $a_{1,n}$, and the third by $a_{2,n}$. Then, by distinguishing whether the center belongs to the independent subset or not, we obtain the following system of recurrence equations:

$$\begin{aligned}
a_{0,n+1} &= a_{0,n}^2(a_{0,n} + a_{1,n})^{q-2} + a_{1,n}^2(a_{1,n} + a_{2,n})^{q-2},\\
a_{1,n+1} &= a_{0,n}a_{1,n}(a_{0,n} + a_{1,n})^{q-2} + a_{1,n}a_{2,n}(a_{1,n} + a_{2,n})^{q-2},\\
a_{2,n+1} &= a_{1,n}^2(a_{0,n} + a_{1,n})^{q-2} + a_{2,n}^2(a_{1,n} + a_{2,n})^{q-2}.
\end{aligned}$$

We are interested in the total quantity

$$a_n = a_{0,n} + 2a_{1,n} + a_{2,n},$$

which, by the recurrence equations given above, satisfies

$$a_{n+1} = (a_{0,n} + a_{1,n})^q + (a_{1,n} + a_{2,n})^q.$$

Taking $x_n = \log a_n$, we obtain the recurrence

$$x_{n+1} = qx_n + \log\left(\left(\frac{a_{0,n} + a_{1,n}}{a_n}\right)^q + \left(\frac{a_{1,n} + a_{2,n}}{a_n}\right)^q\right).$$

Denote by d_n the second summand, which can be estimated easily by using the fact that

$$\frac{a_{0,n} + a_{1,n}}{a_n} + \frac{a_{1,n} + a_{2,n}}{a_n} = 1$$

and $x \mapsto x^q$ is a convex function: we have $0 \geq d_n \geq \log 2^{1-q} = (1-q)\log 2$. Therefore, d_n is bounded. Now, the solution of

$$x_{n+1} = qx_n + d_n$$

is given by

$$x_n = q^n \left(x_0 + \frac{d_0}{q} + \frac{d_1}{q^2} + \cdots + \frac{d_{n-1}}{q^n} \right).$$

Since d_n is bounded, the sum

$$\sum_{k=0}^{\infty} \frac{d_k}{q^{k+1}}$$

converges, so x_n can be written as

$$x_n = q^n \left(x_0 + \sum_{k=0}^{\infty} \frac{d_k}{q^{k+1}} + R(n) \right),$$

and $R(n)$ satisfies $0 \leq R(n) \leq q^{-n} \log 2$, which means that $a_n = C(n)\alpha_{p,q}^{q^n}$, where $1 \leq C(n) \leq 2$ and $\alpha_{p,q}$ is given by

$$\alpha_{p,q} = \exp\left(x_0 + \sum_{k=0}^{\infty} \frac{d_k}{q^{k+1}} \right).$$

By calculating the first values of $a_{0,n}, a_{1,n}$ and $a_{2,n}$ explicitly (the starting values are $a_{0,0} = p-1, a_{1,0} = 1, a_{2,0} = 0$), we obtain an estimate for $\alpha_{p,q}$ – breaking up with d_1 gives the stated result. Furthermore, we see that

$$\alpha_{p,q} = \lim_{n\to\infty} a_n^{q^{-n}}$$

is uniformly convergent in p (since the error term can be bounded independently of p as above). Therefore, we can also obtain the Laurent expansion of $\alpha_{p,q}$ at $p = \infty$ by calculating the expansions of $a_n^{q^{-n}}$ (it is easy to see that the coefficients in the Laurent series of a_n must satisfy some linear recurrence equations) and passing to the limit. We obtain

$$\alpha_{p,q} = p - \frac{1}{q-1} + \frac{2-q}{2(q-1)^2 p} + O(p^{-2})$$

for $q > 2$. In the simple case of $q = 2$, the expansion is

$$p - 1 + p^{-1} + p^{-2} - 2p^{-4} - 3p^{-5} + p^{-6} + 11p^{-7} + 15p^{-8} - 13p^{-9} - 77p^{-10} - 86p^{-11} + \cdots$$

which belongs to the function

$$\alpha_{p,2} = \frac{1}{2}\left(-1 + p + \sqrt{5 - 2p + p^2} \right).$$

In this special case, the corresponding graphs are chains of K_p's, and everything can be reduced to linear recurrence equations. It is not easy to tell whether $\alpha_{p,q}$ can be expressed by elementary functions in general. For $q = 3$, for instance, we obtain

$$p - \frac{1}{2} - \frac{1}{8}p^{-1} + \frac{7}{16}p^{-2} + \frac{91}{128}p^{-3} + \frac{827}{768}p^{-4} + \frac{2657}{3072}p^{-5} - \frac{3547}{6144}p^{-6} - \frac{138861}{32768}p^{-7} + \cdots$$

Last, we prove that $C(n)$ tends to a limit. We note first that the quantities

$$u_n = \frac{a_{0,n}}{a_{1,n}} \qquad \text{and} \qquad v_n = \frac{a_{2,n}}{a_{1,n}}$$

are bounded: trivially, $a_{1,n} \leq a_{0,n}$. Moreover, for each independent subset that doesn't contain $\phi_{1,n}(1)$, we obtain an independent subset containing $\phi_{1,n}(1)$ by removing the neighbors of $\phi_{1,n}(1)$ (there is at most one of them within an independent set since they are pairwise adjacent) and adding

$\phi_{1,n}(1)$. This shows that $a_{0,n} \le pa_{1,n}$ (as an a-priori estimate). Analogously, $a_{2,n} \le a_{1,n} \le pa_{2,n}$ holds as well. Then we observe that

$$\begin{aligned}
|u_{n+1}v_{n+1} - 1| &= \left| \frac{(1+u_n)^{q-2}(1+v_n)^{q-2}(u_nv_n-1)^2}{(u_n(1+u_n)^{q-2} + v_n(1+v_n)^{q-2})^2} \right| \\
&\le \left| \frac{(1+u_n)^{q-2}(1+v_n)^{q-2}(u_nv_n-1)^2}{u_n^2(1+u_n)^{2(q-2)}} \right| \\
&= \left(\frac{1+v_n}{1+u_n} \right)^{q-2} \frac{(u_nv_n-1)^2}{u_n^2} \\
&\le |u_nv_n - 1| \cdot \left| \frac{u_nv_n-1}{u_n^2} \right|
\end{aligned}$$

We note that $\frac{1}{p} - 1 \le u_nv_n - 1 \le u_n^2 - 1$ and $1 \le u_n \le p$ by our a-priori-estimates. Therefore,

$$\left| \frac{u_nv_n-1}{u_n^2} \right| \le 1 - \frac{1}{p^2}$$

for all n, which shows that u_nv_n tends to 1 as $n \to \infty$.

This means that the dynamical system

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_0^2(a_0+a_1)^{q-2} + a_1^2(a_1+a_2)^{q-2} \\ a_0a_1(a_0+a_1)^{q-2} + a_1a_2(a_1+a_2)^{q-2} \\ a_1^2(a_0+a_1)^{q-2} + a_2^2(a_1+a_2)^{q-2} \end{pmatrix},$$

regarded as a map in projective space, has a set of (super-attractive, which is easy to verify) fixed points given by the algebraic curve $\{(z, 1, \frac{1}{z}) : z \in \mathbb{C}\}$, and the vector $(a_{0,n}, a_{1,n}, a_{2,n})$ has to tend to a fixed point from this set. The parameterization of the curve shows that the percentage of independent subsets which contain one of the distinguished vertices is asymptotically independent of the other.

Now, we conclude that d_n tends to a limit d, which, in turn, means that

$$q^n R(n) = -q^n \sum_{k=n}^{\infty} \frac{d_k}{q^{k+1}}$$

tends to $-\frac{d}{q-1}$. This gives us the constant term in the asymptotics of a_n. ∎

5.3. **Antichains in trees with finitely many cone types.** In this section, we will regard a rooted tree of the type described in Example 3 of Section 2 as a partially ordered set and count the number of antichains in a tree of this type. In particular, we will prove the following theorem:

Theorem 6. *Let $A = (a_{ij})$ be an $m \times m$ matrix with non-negative integer entries, and define $X_{k,n}$ as in Example 3 of Section 2. Let $c_{k,n}$ be the number of antichains in $X_{k,n}$. Then we have*

$$c_{k,n} \sim \exp\left(\sum_{s=1}^{S} \lambda_s^n P_{s,k}(n) + Q_k(n) \log n + R_k(n) \right). \tag{4}$$

Here, the λ_s denote the eigenvalues of A of absolute value not less than 1 (except 1), and $P_{s,k}$, Q_k and R_k are computable polynomials. Moreover, Q_k is identically 0 unless $\lambda = 1$ is an eigenvalue of A.

Proof. It is easy to see that

$$c_{k,n+1} = \prod_{j=1}^{s(k)} (1 + c_{\tau_k(j),n}) = \prod_{i=1}^{m} (1 + c_{i,n})^{a_{ki}}, \tag{5}$$

since an antichain in $X_{k,n}$ induces antichains (or empty sets) in the parts of $X_{k,n}$. If we substitute $x_{k,n} = \log c_{k,n}$, we obtain

$$x_{k,n+1} = \sum_{i=1}^{m} a_{ki} x_{i,n} + \sum_{i=1}^{m} a_{ki} \log(1 + c_{i,n}^{-1}).$$

Now we need an a-priori estimate for $c_{k,n}$. We prove that $c_{k,n}$ is either a non-constant polynomial in n for $n > n_0$ or grows at least exponentially. The former is only the case when the cones belonging to the vertices of $X_{k,n}$ are – with only finitely many exceptions – linear chains. We show this by considering the number of leaves of $X_{k,n}$. This number is given by a linear recursion (depending on A) and is non-decreasing. Thus it is either bounded (which means that almost all cones are linear chains) or grows at least linearly. Note that any collection of leaves forms an antichain, and that the number of antichains in a linear chain of length n is exactly n. Together with (5), this implies that $c_{k,n}$ is a polynomial in n for all $n > n_0$ if it does not grow at least exponentially.

We write $\mathbf{x}_n$ for the column vector $(x_{1,n}, \dots, x_{m,n})^{\mathrm{t}}$ and $\mathbf{d}_n = (d_{1,n}, \dots, d_{m,n})^{\mathrm{t}}$, where $d_{k,n} = \log(1 + c_{k,n}^{-1})$. Then the recursion transforms to

$$\mathbf{x}_{n+1} = A\mathbf{x}_n + A\mathbf{d}_n$$

or

$$\mathbf{x}_n = A^n\mathbf{x}_0 + A^n\mathbf{d}_0 + A^{n-1}\mathbf{d}_1 + \cdots + A\mathbf{d}_{n-1}.$$

Now, let $S^{-1}TS$ be the Jordan decomposition of A. Then this can be rewritten as

$$\mathbf{x}_n = S^{-1}(T^nS\mathbf{x}_0 + T^nS\mathbf{d}_0 + T^{n-1}S\mathbf{d}_1 + \cdots + TS\mathbf{d}_{n-1}).$$

For the inner sum, we may suppose that T is a single Jordan block. The total vector is then obtained from joining the vectors belonging to the single Jordan blocks. Let λ be the eigenvalue the Jordan block T belongs to and t the size of the block. We distinguish the following three cases:

(1) $|\lambda| < 1$: Then, since $A^j = O(\lambda^j j^{t-1})$ and $\mathbf{d}_j = O(j^{-1})$, we have

$$T^nS\mathbf{x}_0 + T^nS\mathbf{d}_0 + \cdots + TS\mathbf{d}_{n-1} = O\big(\tfrac{1}{n}\big).$$

(2) $|\lambda| > 1$: T is an invertible matrix, so we can write

$$\begin{aligned} T^nS\mathbf{x}_0 + T^nS\mathbf{d}_0 + \cdots + TS\mathbf{d}_{n-1} &= T^n\left(S\mathbf{x}_0 + \sum_{j=0}^{n-1} T^{-j}S\mathbf{d}_j\right) \\ &= T^n\left(S\mathbf{x}_0 + \sum_{j=0}^{\infty} T^{-j}S\mathbf{d}_j - \sum_{j=n}^{\infty} T^{-j}S\mathbf{d}_j\right). \end{aligned}$$

The infinite sums are convergent, since $T^{-j} = O(\lambda^{-j}j^{t-1})$. For $j > n_0$, we know that all $d_{k,j}$ are either of the form $\log\big(1 + p(j)^{-1}\big)$ for some polynomial p or exponentially decreasing in terms of j. By using the expansion around ∞, we obtain

$$S\mathbf{d}_j = (p_1(j^{-1}), \dots, p_t(j^{-1}))^{\mathrm{t}} + O(j^{-t}),$$

where the p_i are polynomials of degree $\le t-1$ with constant coefficient 0. It is well known that

$$\sum_{j=n}^{\infty} \lambda^{-j} j^{\ell} = \lambda^{-n}\left(\sum_{\nu=0}^{s-1}\binom{\ell}{\nu}\operatorname{Li}_{-\nu}(\lambda^{-1})n^{\ell-\nu} + O(n^{\ell-s})\right),$$

where $\operatorname{Li}_\sigma(z) = \sum_{j=0}^{\infty} j^{-\sigma}z^j$ is a *polylogarithm*, see [23]. Therefore, the sum $\sum_{j=n}^{\infty} T^{-j}S\mathbf{d}_j$ can be written in the form

$$T^{-n}\cdot\big((r_1(n), \dots, r_t(n))^{\mathrm{t}} + O(n^{-1})\big),$$

where the r_i are polynomials of degree $\le t-1$. Altogether, this implies that

$$T^n\sum_{j=n}^{\infty} T^{-j}S\mathbf{d}_j = \mathbf{R}(n) + O(n^{-1}),$$

where $\mathbf{R}$ is a vector of polynomials of degree $\le t-1$.

(3) $|\lambda| = 1$: this case is almost analogous to $|\lambda| > 1$. Again, we expand $\mathbf{d}_j$ around ∞; then, consider the sums

$$\sum_{j=1}^{n-1} \lambda^{-j} j^{\ell}.$$

For $\ell \leq -t$, these sums are convergent with an error term of $\sum_{j=n}^{\infty} \lambda^{-j} j^{\ell} = O(n^{-t})$. For all other ℓ, these sums can be written as

$$\sum_{j=1}^{n-1} \lambda^{-j} j^{\ell} = C_{\ell} + \lambda^{-n} L(n) + O(n^{-r}),$$

where r can be made arbitrary and $L(n)$ is an expansion around $n = \infty$. This yields terms of the form

$$T^n \sum_{j=0}^{n} T^{-j} S \mathbf{d}_j = \lambda^n \mathbf{P}(n) + \mathbf{R}(n) + O(n^{-1})$$

for some polynomials $\mathbf{P}, \mathbf{Q}$ of degree $\leq t-1$. The only exception is $\lambda = 1$ – here, logarithmic terms may appear in view of $\sum_{j=1}^{n} j^{-1} \sim \log n$.

Altogether, we obtain a formula of the type (4), which finishes the proof. ∎

Remark. Note that the same way of reasoning can be used for maximal antichains, whose recursion is given by

$$c_{k,n+1} = 1 + \prod_{j=1}^{s(k)} c_{\tau_k(j),n} = 1 + \prod_{i=1}^{m} c_{i,n}^{a_{ki}},$$

which transforms into the recursion for antichains after performing the simple substitution $c_{k,n} = 1 + \tilde{c}_{k,n}$.

Remark. The number of antichains is also the number of subtrees containing the root – the leaves of such a subtree always define an antichain and vice versa.

We give two particularly nice examples for our theorem:

Example 1. Consider the complete binary tree belonging to the 1×1-matrix with a single entry of 2. Then the number c_n of antichains is given by $c_0 = 1$ and $c_{n+1} = (c_n + 1)^2$. The solution of this recursion has already been given by Aho and Sloane [1]; this was also noted by Székely and Wang [37] who considered the number of subtrees in a complete binary tree. In fact, we have

$$c_n = \lfloor \alpha^{2^n} \rfloor - 1,$$

where

$$\alpha = \exp\left(\sum_{i=0}^{\infty} 2^{-i} \log(1 + c_i^{-1}) \right) = 2.258518\ldots$$

The sequence $(c_n) = (1, 4, 25, 676, 458329, \ldots)$ is number A004019 in Sloane's "On-Line Encyclopedia of Integer Sequences" [36].

Example 2. Let $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then we obtain a comb-like tree. The corresponding recursion is given by $c_0 = 1$ and

$$c_n = (n+1)(c_{n-1} + 1).$$

This sequence is known for more than 300 years, counting the number of permutations of nonempty subsets of $\{0, 1, \ldots, n\}$. The solution of the recursion is seen to be $\lfloor e(n+1)! \rfloor - 1$ (Sloane's A007526 [36]; the first terms of the sequence are $1, 4, 15, 64, 325, 1956, 13699, \ldots$).

5.4. **Connected subsets in a Sierpiński graph.** The construction of Sierpiński graphs was described in Example 4 of Section 2. We will consider the case $d = 2$ only and calculate the number of connected subsets in the level-n Sierpiński graph. A vertex subset of a graph is connected, if it induces a connected subgraph.

This example shows us that it may be necessary to consider several auxiliary properties as well. In fact, we need seven different sequences: we consider sets of vertices with the property that every connected component of the induced subgraphs contains at least one of the corner vertices. Our auxiliary sequences are distinguished by the number of corner vertices contained in the subsets and the partition of these corner vertices induced by the connected components.

- $a_{1,n}$ counts the number of subsets with three connected components, each of which contains one corner vertex.
- $a_{2,n}$ counts the number of subsets with two connected components, one of them containing two corner vertices, the other component one,
- $a_{3,n}$ counts the number of subsets with two connected components, each of which contains one corner vertex.
- $a_{4,n}$ counts the number of connected subsets containing all corner vertices,
- $a_{5,n}$ counts the number of connected subsets containing two corner vertices.
- $a_{6,n}$ counts the number of connected subsets containing one corner vertex.
- Eventually, $a_{7,n}$ counts the number of connected subsets containing no corner vertex (excluding the empty set).

It takes some time and patience to work through all possibilities and thus determine the correct recurrence equations, but this task can be simplified by means of a computer. As an example, we derive the equation for $a_{4,n+1}$. Let the three vertices which connect the parts of a Sierpiński graph be called the *links*. At least two of them have to belong to a connected set containing all corner vertices – otherwise, it is impossible to connect the corners.

- If all links are contained in a connected subset, either all the induced subsets in all three parts are connected or two of them are connected and one of them has two connected components, each containing one of the links. The corner can be contained in either of these components. This yields a summand of $a_{4,n}^3 + 6a_{2,n}a_{4,n}^2$.
- Suppose that only two of the links are contained in a connected subset. Then the induced subsets in all the parts have to be connected, which leads to a summand of $3a_{4,n}a_{5,n}^2$.

So we arrive at the recursive relation

$$a_{4,n+1} = a_{4,n}^3 + 6a_{2,n}a_{4,n}^2 + 3a_{4,n}a_{5,n}^2.$$

In a similar way, recurrence equations can be determined in all other cases as well by accurately distinguishing cases. This leads us to the following system (note that the polynomials on the right are not homogeneous; however, we could achieve this by introducing the trivial sequence which counts the empty set only):

$$\begin{aligned}
a_{1,n+1} &= 12a_{1,n}a_{2,n}a_{4,n} + 3a_{1,n}a_{5,n}^2 + 14a_{2,n}^3 + 12a_{2,n}a_{3,n}a_{5,n} \\
&\quad + 3a_{2,n}^2a_{4,n} + 3a_{3,n}^2a_{4,n} + 6a_{3,n}a_{5,n}a_{6,n} + a_{6,n}^3, \\
a_{2,n+1} &= a_{1,n}a_{4,n}^2 + 7a_{2,n}^2a_{4,n} + a_{2,n}a_{4,n}^2 + 3a_{2,n}a_{5,n}^2 + 2a_{3,n}a_{4,n}a_{5,n} + a_{5,n}^2a_{6,n}, \\
a_{3,n+1} &= 2a_{1,n}a_{4,n}a_{5,n} + 4a_{2,n}a_{3,n}a_{4,n} + 2a_{2,n}a_{4,n}a_{5,n} + 6a_{2,n}^2a_{5,n} + 4a_{2,n}a_{5,n}a_{6,n} \\
&\quad + 2a_{3,n}a_{4,n}a_{6,n} + 3a_{3,n}a_{5,n}^2 + 2a_{3,n}a_{5,n} + 2a_{5,n}a_{6,n}^2 + a_{6,n}^2, \\
a_{4,n+1} &= 6a_{2,n}a_{4,n}^2 + a_{4,n}^3 + 3a_{4,n}a_{5,n}^2, \\
a_{5,n+1} &= 4a_{2,n}a_{4,n}a_{5,n} + a_{3,n}a_{4,n}^2 + a_{4,n}^2a_{5,n} + 2a_{4,n}a_{5,n}a_{6,n} + a_{5,n}^3 + a_{5,n}^2, \\
a_{6,n+1} &= 2a_{2,n}a_{5,n}^2 + 2a_{3,n}a_{4,n}a_{5,n} + a_{4,n}a_{5,n}^2 + a_{4,n}a_{6,n}^2 \\
&\quad + 2a_{5,n}^2a_{6,n} + 2a_{5,n}a_{6,n} + a_{6,n}, \\
a_{7,n+1} &= 3a_{3,n}a_{5,n}^2 + a_{5,n}^3 + 3a_{5,n}a_{6,n}^2 + 3a_{6,n}^2 + 3a_{7,n}.
\end{aligned} \tag{6}$$

The initial values are $(a_{1,0}, a_{2,0}, a_{3,0}, a_{4,0}, a_{5,0}, a_{6,0}, a_{7,0}) = (0, 0, 0, 1, 1, 1, 0)$, and the total number of connected subsets (including the empty set) at level n is given by $a_{4,n} + 3a_{5,n} + 3a_{6,n} + a_{7,n} + 1$.

Asymptotically, the terms of total degree three in our system of recurrences are much larger than the others, so we have to study the dynamical system generated by these terms:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} \mapsto \begin{pmatrix} 12a_1a_2a_4 + 3a_1a_5^2 + 14a_2^3 + 12a_2a_3a_5 + 3a_2^2a_4 + 3a_3^2a_4 + 6a_3a_5a_6 + a_6^3 \\ a_1a_4^2 + 7a_2^2a_4 + a_2a_4^2 + 3a_2a_5^2 + 2a_3a_4a_5 + a_5^2a_6 \\ 2a_1a_4a_5 + 4a_2a_3a_4 + 2a_2a_4a_5 + 6a_2^2a_5 + 4a_2a_5a_6 + 2a_3a_4a_6 + 3a_3a_5^2 + 2a_5a_6^2 \\ 6a_2a_4^2 + a_4^3 + 3a_4a_5^2 \\ 4a_2a_4a_5 + a_3a_4^2 + a_4^2a_5 + 2a_4a_5a_6 + a_5^3 \\ 2a_2a_5^2 + 2a_3a_4a_5 + a_4a_5^2 + a_4a_6^2 + 2a_5^2a_6 \\ 3a_3a_5^2 + a_5^3 + 3a_5a_6^2 \end{pmatrix}$$

Unfortunately, it has no positive fixed points in projective space. So we have to apply a little trick: set $\gamma = \frac{5}{3}$ and

$$a_{i,n} = \begin{cases} \gamma^{3n/2}A_{i,n} & \text{for } i = 1, \\ \gamma^{n/2}A_{i,n} & \text{for } i = 2, 3, \\ \gamma^{-n/2}A_{i,n} & \text{for } i = 4, 5, 6, 7. \end{cases}$$

In addition, we denote by $\mathbf{A}_n$ the vector $(A_{1,n}, \ldots, A_{7,n})$. Then, our recurrence equations transform to

$$\mathbf{A}_{n+1} = \mathbf{P}(\mathbf{A}_n) + \gamma^{-n}\mathbf{Q}_1(\mathbf{A}_n) + \gamma^{-2n}\mathbf{Q}_2(\mathbf{A}_n) + \gamma^{-3n}\mathbf{Q}_3(\mathbf{A}_n) + \mathbf{R}(\mathbf{A}_n),$$

where $\mathbf{P}$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ are homogeneous polynomials of degree three and $\mathbf{R}$ contains the remaining terms of lower degree. The polynomial $\mathbf{P}$ is given by

$$\mathbf{P} : \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{pmatrix} \mapsto \begin{pmatrix} \gamma^{-3/2}(14A_2^3 + 12A_1A_2A_4) \\ \gamma^{-1/2}(7A_2^2A_4 + A_1A_4^2) \\ \gamma^{-1/2}(4A_2A_3A_4 + 6A_2^2A_5 + 2A_1A_4A_5) \\ \gamma^{1/2}(6A_2A_4^2) \\ \gamma^{1/2}(A_3A_4^2 + 4A_2A_4A_5) \\ \gamma^{1/2}(2A_2A_5^2 + 2A_3A_4A_5) \\ \gamma^{1/2}(3A_3A_5^2) \end{pmatrix}.$$

When we study the dynamical system generated by $\mathbf{P}$ we observe that the algebraic surface defined by

$$\left\{(A_1, A_2, A_3, A_4, A_5, A_6, A_7) = \left(\frac{1}{10\mu^3}, \frac{1}{2\sqrt{15}\mu}, \frac{\lambda}{\sqrt{15}\mu}, \mu, \lambda\mu, \lambda^2\mu, \lambda^3\mu\right)\right\}$$

is the set of attractive fixed points. Indeed, one can check that, as a vector in projective space, $\mathbf{A}_n$ tends to a fixed point $\mathbf{C}$, whose numerical value is

$$(0.573118, 0.291082, 0.817477, 0.443515, 0.622786, 0.874517, 1.228000).$$

A rigorous proof of this fact would involve the following steps:

- Check that $\mathbf{A}_n$ lies within a suitable neighborhood of the fixed point for a sufficiently large value of n,
- prove inductively that it will stay within these boundaries for all larger n (since the fixed point is attractive, the first term is a contraction within a suitable neighborhood; the remaining terms are easily estimated since they are very small for sufficiently large n).

Again, we notice an "independency phenomenon" for the corners of the triangle.

So, finally, we obtain the asymptotics $\mathbf{A}_n \sim \mathbf{C} \cdot \beta^{3^n}$, where the numerical value of β is 2.3032106556. Altogether, we find the asymptotic number of connected subsets in the level n-Sierpiński graph to be

$$6.163424 \cdot \gamma^{-\frac{n}{2}} \cdot \beta^{3^n} \sim 2.940541 \cdot V^{\frac{1}{2}\left(1-\frac{\log 5}{\log 3}\right)} \cdot \beta^{\frac{2V}{3}}.$$

Here $V = \frac{3}{2}(3^n + 1)$ denotes the number of vertices in this formula. The numerical value of $\beta^{2/3}$ is 1.7440373203... The first terms of the sequence are

$$8, 48, 6307, 16719440488, 484190291407629184897238968931, \ldots$$

Remark. We observe that there are more sets of the "half-connected" (two connected components) or "one-third-connected" (three connected components) type – by an exponential factor – than connected sets. Furthermore, when we take a closer look at the recurrence relations, we see that the summands which contribute most always correspond to the case that all three "links" are contained in the set. This means that almost all (in some sense) connected subsets contain all three links.

Remark. The Sierpiński triangle is easily generalized by varying the number of subdivisions per triangle side (and thus varying the number of subtriangles) or generalizing the construction to higher dimensions. By computer experiments, we observed that similar results hold for these generalizations, where the constant $\gamma = \frac{5}{3}$ is replaced by other rational numbers. For the 3- and 4-dimensional analogues, the constants are $\frac{3}{2}$ and $\frac{7}{5}$ respectively, leading us to the conjectured formula $\frac{d+3}{d+1}$, where d is the dimension. On the other hand, by increasing the number of subdivisions, we obtain the sequence

$$\frac{5}{3}, \frac{15}{7}, \frac{103}{41}, \frac{1663}{591}, \frac{21559}{7025}, \ldots$$

which is rather difficult to explain. We did not find any hints on its origin in Sloane's encyclopedia [36]. It is a remarkable fact, however, that all these constants are indeed rational numbers, since they are given by rather complicated algebraic systems of equations only. It seems to be a highly challenging problem to find a proof for this.

Surprisingly, these rational numbers exactly match the resistance scaling factors of the generalized Sierpiński gaskets, see [3] for a definition of this constant: Consider the level-n graph $X_{1,n}$ as electrical network with constant resistant on its edges and denote by $\mathcal{E}_n$ the associated energy form. Then there is a restriction of $\mathcal{E}_{n+1}$ to $X_{1,n}$, which is called the trace of $\mathcal{E}_{n+1}$. It turns out that $\mathcal{E}_n$ and the trace of $\mathcal{E}_{n+1}$ are the same up to a constant, which is called the resistance scaling factor.

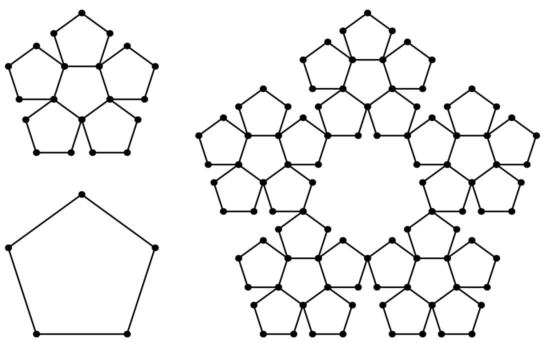

FIGURE 6. The Pentagasket: A pentagonal analogue of the Sierpiński gasket.

Interestingly, when we consider the pentagonal analogue of the Sierpiński gasket, see Figure 6, the basis of the exponential factor is not rational any more; however, it still equals the resistance scaling factor, which is $\frac{1}{10}(9 + \sqrt{161})$ in this case, see for example [12].

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