AN ELEMENTARY PROOF OF AN INEQUALITY FOR CONVEX LATTICE POLYGONS

STEPHAN WAGNER

Abstract. Bárány and Tokushige solved the problem of characterizing the asymptotic behavior of the minimum area $a(n)$ of a convex lattice n-gon: they showed that the limit of $a(n)/n^3$ exists and that it is most probably close to $0.0185067\ldots$. In this note, a short and elementary proof is given for the fact that $a(n) \geq n^3(1/72 + o(1))$, which is a weaker result than that of Bárány and Tokushige, but improves on previous elementary proofs due to Rabinowitz and Cai.

1. Introduction

The minimal area of an arbitrary lattice polygon with n vertices is easily determined (using Picks Theorem) as $(n-2)/2$; however, for convex lattice n-gons, the area is bounded below by cn^3 with some universal constant c. For this fact, different proofs have been given by G.E. Andrews [1], V.I. Arnol'd [2], W. Schmidt [10], Bárány/Pach [3] and T.-X. Cai [6].

In the following, $a(n)$ denotes the minimum area of a convex lattice n-gon. Simpson [11] provides a nice geometric proof of the following characterization:

Lemma 1 (Simpson 1990)**.**

$$a(2n) = \min \sum_{i=1}^{n} \sum_{j=i+1}^{n} (y_i x_j - x_i y_j),$$

where the minimum is taken over all sequences $(x_i, y_i)_{1 \leq i \leq n}$ *of ordered pairs satisfying the following four conditions:*

- $y_i x_j - x_i y_j > 0$ *for all* $1 \leq i < j \leq n$,
- $\gcd(x_i, y_i) = 1$ *for all* $1 \leq i \leq n$,
- $(x_1, y_1) = (0, 1)$,
- $y_i \geq x_i > 0$ *for all* $2 \leq i \leq n$.

In the following, we will call a sequence that satisfies Simpson's conditions *admissible.* Furthermore, Simpson shows that

$$\left\lfloor \frac{a(2n+2) + a(2n)}{2} \right\rfloor + \frac{1}{2} \leq a(2n+1) \leq a(2n+2) - \frac{1}{2}.$$

Thus it is sufficient to investigate the case of even n. The first few values of $a(n)$ are given in the following table (see [11]):

n	3	4	5	6	7	8	9	10	12	14	16	18
$a(n)$	0.5	1	2.5	3	6.5	7	10.5	14	24	40	59	87

As Bárány and Tokushige [4] point out, the problem is equivalent to finding an 0-symmetric body of minimum area that contains n primitive vectors (i.e., the coordinates are coprime). Therefore, an upper bound for $a(n)$ can easily be given using a circle of appropriate radius in this equivalent problem, yielding the bound $n^3(1+o(1))/54$ (see [4]). They also show that the constant $\frac{1}{54}$ is pretty close to the true value of $\lim_{n\to\infty} \frac{a(n)}{n^3}$ (which they prove to exist): they are able to reduce the problem to finitely many optimization problems, and their calculations suggest that the correct value is approximately $0.0185067\ldots$ (compare to $\frac{1}{54} = 0.0185185\ldots$), thus solving the problem almost completely.

Up to that point, the best known lower bound $n^3/(8\pi^2)$ had been given by S. Rabinowitz [9]; T.-X. Cai [5, 6] uses Simpson's characterization to give an elementary proof of the inequality $a(n) \geq \frac{n^3}{1152} + O(n^2)$. Before that, estimates have also been provided by Rabinowitz [8] and Colbourn and Simpson [7].

The result that is proved in this paper is the following:

Theorem 1.

$$a(n) \geq \frac{n^3}{72} + O(n^{5/2}).$$

Of course, this result is much weaker than that of Bárány and Tokushige, but the proof is short and elementary, and so is appears to be interesting on its own right.

2. Proof of the main theorem

Let $(x_i, y_i)_{1\leq i\leq n}$ be any admissible sequence. Choose a $k \geq 2$ such that y_k is maximal. Then $(x_i, y_i)_{\substack{1\leq i\leq n\\ i\neq k}}$ is (trivially) admissible again. So we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=i+1}^{n}(y_ix_j - x_iy_j) &= \sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=i+1\\ j\neq k}}^{n}(y_ix_j - x_iy_j) + \sum_{i=1}^{n}(y_ix_k - x_iy_k)\\ &\geq a(2n-2) + \sum_{i=1}^{n}|y_ix_k - x_iy_k|\end{aligned}$$

and by taking the minimum

$$a(2n) \geq a(2n-2) + \sum_{i=1}^{n}|y_ix_k - x_iy_k|.$$

Now consider the last sum: by the first admissibility condition, none of the summands can be 0. We show that there is at most one index i with $y_ix_k - x_iy_k = c$ for any other $c \in \mathbb{Z}$:

If $y_ix_k - x_iy_k = c$, then we must have $y_ix_k \equiv c \mod y_k$. As x_k and y_k are coprime, the solution of $zx_k \equiv c \mod y_k$ is unique modulo y_k. As $0 < y_i \leq y_k$ by the maximality of y_k, y_i is uniquely determined by $y_ix_k - x_iy_k = c$ (and so is x_i from $x_i = \frac{y_ix_k - c}{y_k}$). Thus there can be at most one i for which we have $y_ix_k - x_iy_k = c$. So $|y_ix_k - x_iy_k| = b$ for at most two indices i for any $b \in \mathbb{N}$.

Now let $b = 2d \in \mathbb{N}$ be even. We claim that $|y_ix_k - x_iy_k| = b$ for at most one i (not two), if $y_k \nmid d$.

To prove this, let z_1 und z_2 be the solutions of $zx_k \equiv d \mod y_k$ and $zx_k \equiv -d \mod y_k$, respectively, such that $1 \leq z_1, z_2 \leq y_k$ holds. As $y_k \nmid d$, z_1 and z_2 are not equal y_k. Clearly $z_1 \equiv -z_2 \mod y_k$ and thus $z_1 + z_2 = y_k$. So either $z_1 \leq y_k/2$ or $z_2 \leq y_k/2$. Assume that the former holds. Then $2z_1$ is a solution of the equation $zx_k \equiv 2d \mod y_k$. Choose u_1 in such a way that $z_1x_k - u_1y_k = d$. Then the pair $(2z_1, 2u_1)$ satisfies $2z_1x_k - 2u_1y_k = 2d = b$ and $2z_1 \leq y_k$. But this cannot be a pair (y_i, x_i), as $2z_1$ und $2u_1$ are not coprime. So one of the possible solutions of $|y_ix_k - x_iy_k| = b$ cannot be attained by a pair (y_i, x_i), which proves the claim (the second case is proved analogously).

Thus we immediately obtain an estimate for the sum $\sum_{i=1}^{n}|y_ix_k - x_iy_k|$: it may contain every odd number at most twice and every even number that is not divisible by $2y_k$ at most once. Consequently, we get

$$\sum_{i=1}^{n} |y_i x_k - x_i y_k| \geq 1 + 1 + 2 + 3 + 3 + \ldots + \left\lfloor \frac{2R-1}{3} \right\rfloor + 2y_k + 4y_k + \ldots + 2Sy_k,$$

where R and S are taken in such a way that the number $R-1+S$ of summands equals $n-1$ and that $2Sy_k \leq \lfloor \frac{2R-1}{3} \rfloor$.

As all pairs (x_i, y_i) must be different with $x_i \leq y_i$, there are at most two pairs with $y_i = 1$, three pairs with $y_i = 2$, and so on. Therefore we have $\sum_{j=1}^{y_k}(j+1) = y_k(y_k+3)/2 \geq n$ and it follows that $y_k \geq \sqrt{2n+9/4} - 3/2$. Now we have the following estimate for R and S:

$$2Sy_k \leq \left\lfloor \frac{2R-1}{3} \right\rfloor \leq \frac{2R}{3} \qquad \text{or} \qquad S \leq \frac{R}{3y_k}.$$

From $R + S = n$ we can conclude now that $n \leq R\left(1 + \frac{1}{3y_k}\right)$ and thus

$$R \geq n\left(1 - \frac{1}{3y_k+1}\right) \geq n\left(1 - \frac{1}{3\sqrt{2n+9/4} - 7/2}\right) = n + O(\sqrt{n}).$$

The sum

$$1 + 1 + 2 + 3 + 3 + 4 + \ldots + \left\lfloor \frac{2R-1}{3} \right\rfloor$$

is evaluated easily:

$$\sum_{l=2}^{R} \left\lfloor \frac{2l-1}{3} \right\rfloor = \left\lfloor \frac{R^2-R+1}{3} \right\rfloor = \begin{cases} 3r^2 - r & R = 3r \\ 3r^2 + r & R = 3r+1 \\ 3r^2 + 3r + 1 & R = 3r+2 \end{cases},$$

and $2y_k \sum_{l=1}^{S} l = y_k S(S+1) = O(RS)$. Thus

$$a(2n) - a(2n-2) \geq \left\lfloor \frac{R^2-R+1}{3} \right\rfloor + O(RS) = \frac{R^2}{3} + O(RS) = \frac{n^2}{3} + O(n\sqrt{n}).$$

By summing over all n, we obtain

$$a(2n) \geq \sum_{m=1}^{n} \left(\frac{m^2}{3} + O(m\sqrt{m})\right) = \frac{n^3}{9} + O(n^{5/2}).$$

It immediately follows that $a(n) \geq \frac{n^3}{72} + O(n^{5/2})$ for even n. But by the fact that $\left\lfloor \frac{a(2n+2)+a(2n)}{2} \right\rfloor + \frac{1}{2} \leq a(2n+1) \leq a(2n+2) - \frac{1}{2}$, this remains true for odd n. □

Remark. Explicit computation of the estimates above yields sharp results for small values of n, specifically for $n = 4, 6, 8, 10, 12$.

Remark. The argument for even values of b can be heuristically extended: for instance, consider solutions of $zx_k \equiv \pm d \mod y_k$ satisfying $z_1 \leq y_k/3$ or $z_2 \leq y_k/3$ (if y_k is small compared to n, about two thirds of all values for d satisfy the condition; if y_k is big compared to n, the ratio might be smaller, possibly only one half). These would give solutions of $zx_k \equiv \pm 3d \mod y_k$ with $z \leq y_k$ by simply multiplying by 3. Solutions of that kind cannot be admitted by any pair (x_i, y_i). The same reasoning works for all primes instead of 3.

However, it is not evident at all how to combine the arguments for different primes in an effective way.

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