AN INDEPENDENCE THEOREM FOR THE NUMBER OF MATCHINGS IN GRAPHS

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Abstract. Let $z(G)$ be the number of matchings (independent edge subsets) of a graph G. For a set M of edges and/or vertices, the ratio $r_G(M) = z(G \setminus M)/z(G)$ represents the probability that a randomly picked matching of G does not contain an edge or cover a vertex that is an element of M. We provide estimates for the quotient $r_G(A \cup B)/(r_G(A)r_G(B))$, depending on the sizes of the disjoint sets A and B, their distance and the maximum degree of the underlying graph G. It turns out that this ratio approaches 1 as the distance of A and B tends to ∞, provided that the size of A and B and the maximum degree are bounded, showing asymptotic independence. We also provide an application of this theorem to an asymptotic enumeration problem related to the dimer-monomer model from statistical physics.

1. Introduction and statement of results

The number of matchings (also known as independent edge subsets) of a (finite, simple) graph G, henceforth denoted by $z(G)$, is a parameter that is of relevance, among others, in statistical physics (so-called dimer-monomer model, cf. [5, 8, 9] and other references provided in [3]) and combinatorial chemistry (there, $z(G)$ is known as Hosoya-index of a graph, cf. [7, 10, 11, 17]; more generally, the matching polynomial of a graph is of interest, see [6]). Therefore, the enumeration of matchings has already been investigated for various classes of graphs, most notably trees, hexagonal chains, grid graphs, random graphs and self-similar graphs [2, 3, 12, 13, 14, 16, 18]. In [16], the authors of this short note observed an independence phenomenon for a very particular sequence of graphs. Even though not explicitly stated in this way, a similar observation has been made by Chang and Chen in [3]. Roughly stated, it was proved in both instances that the ratio

$$\frac{z(G_n \setminus \{v_n, w_m\})z(G_n)}{z(G_n \setminus v_n)z(G_n \setminus w_m)} = \frac{\frac{z(G_n \setminus \{v_n, w_m\})}{z(G_n)}}{\frac{z(G_n \setminus v_n)}{z(G_n)} \cdot \frac{z(G_n \setminus w_n)}{z(G_n)}}$$

tended to 1 as the distance of two specific vertices $v_n, w_n \in G_n$ tended to ∞ for a certain sequence $(G_n)_{n \geq 1}$ of graphs. Intuitively, this statement means that the influences of two vertices on the number of matchings are asymptotically independent if the distance between them is large.

The aim of this note is to show that this behavior does not depend on the actual structure of the graphs and that the statement can even be generalized to sets of vertices and/or edges in place of single vertices. For an arbitrary subset A (of vertices and/or edges), the ratio $z(G \setminus A)/z(G)$, i.e. the probability that a randomly selected matching does not contain an edge in A or cover a vertex in A, is denoted by $r_G(A)$. We are able to give an upper and lower estimate for the quotient

$$q_G(A, B) := \frac{r_G(A \cup B)}{r_G(A)r_G(B)} = \frac{r_{G \setminus A}(B)}{r_G(B)} = \frac{z(G \setminus (A \cup B))z(G)}{z(G \setminus A)z(G \setminus B)}$$

and prove that it tends to 1 at an exponential rate as the distance $d(A, B)$ goes to ∞, provided that the sizes of A and B and the maximum degree of G are bounded. Finally, we show how our theorem (which might look somewhat contrived at first sight) can be applied in the context of enumeration problems such as those treated in [3]. The quantity $q_G(A, B)$ can be interpreted in the following two ways:

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- It can be seen as the correlation coefficient of the two events “a randomly selected matching does not intersect A” and “a randomly selected matching does not intersect B”.
- It relates the probability of the event “a randomly selected matching does not intersect B” in the graph G to the probability of the same event in the graph $G \setminus A$.

Our main theorem reads as follows:

Theorem 1. *Let G be a graph and A, B be two disjoint sets of vertices. There are positive constants C and D, $D < 1$, which depend only on the maximum degree $\Delta = \Delta(G)$ of G and the sizes $a = |A|$ and $b = |B|$, such that the inequality*

$$\frac{1}{1 + C\,D^{d-1}} \leq q_G(A, B) \leq 1 + C\,D^{d-1}$$

holds, where $d = d(A, B)$ denotes the distance of A and B. Furthermore, C and D can be chosen as

$$C = (1 + \Delta)^{\min\{a,b\}}$$

and

$$D = 1 - C^{-1}.$$

It is not too difficult to draw the following corollaries from this theorem:

Corollary 2. *Let G be a graph and $A_1, A_2, \ldots, A_k$ be disjoint sets of vertices. Then there are positive constants C and D, $D < 1$, which depend only on the maximum degree $\Delta = \Delta(G)$ of G and the sizes of the A_i such that*

$$(1 + C\,D^{d-1})^{1-k} \leq \frac{r_G\big(\bigcup_{i=1}^{k} A_i\big)}{\prod_{i=1}^{k} r_G(A_i)} \leq (1 + C\,D^{d-1})^{k-1}$$

holds, where $d = \min_{i,j} d(A_i, A_j)$ is the minimal distance between two sets from our collection. C and D can be taken as

$$C = (1 + \Delta)^{\max_i |A_i|}$$

and

$$D = 1 - C^{-1}.$$

Corollary 3. *Let G be a graph and A, B be two disjoint sets of vertices and/or edges such that $d(A, B) \geq 1$. Then the inequality*

$$\frac{1}{(1 + C\,D^{d-1})^4} \leq q_G(A, B) \leq (1 + C\,D^{d-1})^4$$

holds, where $d = d(A, B)$ and C, D can be taken as in Theorem 1.

Of course, it is also possible to combine the two corollaries to obtain a statement for k disjoint sets of vertices and/or edges.

2. Preliminaries

$z(G)$ will always denote the number of matchings (independent edge subsets, i.e. edge sets with the property that no two edges from the set share a common vertex) of a finite, simple graph G, where the empty set is counted as a matching, even if G is potentially an empty graph. Therefore, $z(G)$ is always positive.

Some simple properties of $z(G)$ will be exploited in our proofs, which shall be stated in this section. For completeness, we will provide short proofs for these auxiliary tools as well. First of all, let us state a trivial inequality that will be used throughout the paper without further reference:

Lemma 4. *If H is a subgraph of G (not necessarily induced), then*

$$z(H) \leq z(G),$$

with equality if and only if G can be written as $H \cup \{v_1, v_2, \ldots, v_k\}$, where $v_1, \ldots, v_k$ are isolated vertices.

Proof. Simply note that any matching in H is also a matching in G. Equality can only hold if H has the same set of edges as G (otherwise, there is at least one matching comprising of a single edge that belongs to G, but not to H). ■

Furthermore, we will use two simple reduction techniques: reduction with respect to vertices and reduction with respect to edges. These are generalizations of the well-known formulæ (cf. [7])

$$z(G) = z(G \setminus v) + \sum_{w \sim v} z(G \setminus \{v, w\})$$

(for arbitrary vertices $v \in V(G)$) and

$$z(G) = z(G \setminus e) + z(G \setminus \{v, w\})$$

(for arbitrary edges $e = (v, w)$) and can be deduced from them, but they can also be proved directly without deeper insight:

Lemma 5. *Let G be a graph and $A \subseteq V(G)$ an arbitrary set of vertices. Let $N = N_G[A]$ be the neighborhood of A in G, including A itself (i.e. the set of all vertices v such that the distance $d(v, A)$ is ≤ 1). Furthermore, let $\mathcal{M}_G(A)$ denote the collection of all matchings M (including the empty set) of the induced subgraph $G[N]$ with the property that none of the edges included in M connects two vertices in $N \setminus A$. Finally, for $M \in \mathcal{M}_G(A)$, let $U(M)$ be the set of all endpoints of edges in M. Then we have*

$$z(G) = \sum_{M \in \mathcal{M}_G(A)} z(G \setminus (A \cup U(M))).$$

Proof. For every $M_1 \in \mathcal{M}_G(A)$ and every matching M_2 of $G \setminus (A \cup U(M_1))$, the union $M = M_1 \cup M_2$ is a matching of G (note that the edges in M_1 and the edges in M_2 cannot have any common vertices, since all endpoints of edges in M_1 lie in $U(M_1)$). Conversely, every matching M of G can be split into a matching $M_1 \in \mathcal{M}_G(A)$ (that contains all the edges with endpoints in A) and a matching M_2 of $G \setminus (A \cup U(M_1))$. This readily proves the lemma. ■

Lemma 6. *Let G be a graph and $A \subseteq E(G)$ an arbitrary set of edges. Let $\mathcal{I}(A)$ be the collection of all independent subsets of A, and for $M \in \mathcal{I}(A)$, define $U(M)$ as in Lemma 5. Then we have*

$$z(G) = \sum_{M \in \mathcal{I}(A)} z(G \setminus (A \cup U(M))).$$

Proof. As in the previous proof, every matching M of G can be decomposed into an independent subset $M_1 = A \cap M$ of M and a matching M_2 of $G \setminus (A \cup U(M_1))$ and vice versa, which proves the lemma. ■

3. Proof of Theorem 1

The main theorem is proved by means of induction on the distance d. First of all, we determine an upper and a lower bound that do not depend on the distance. Applying Lemma 5 to A yields

$$q_G(A, B) = \frac{z(G \setminus (A \cup B))z(G)}{z(G \setminus A)z(G \setminus B)} \leq \frac{z(G)}{z(G \setminus A)} = \sum_{M \in \mathcal{M}_G(A)} \frac{z(G \setminus (A \cup U(M)))}{z(G \setminus A)} \leq |\mathcal{M}_G(A)|.$$

We have $|\mathcal{M}_G(A)| \leq (\Delta + 1)^a$, since every element M of $\mathcal{M}_G(A)$ either contains one of the at most Δ edges incident with any vertex $v \in A$ or none of these edges (and no edges which are not incident with a vertex in A at all). Hence,

$$q_G(A, B) \leq (\Delta + 1)^a$$

and by symmetry

$$q_G(A, B) \leq (\Delta + 1)^{\min\{a,b\}}.$$

Note that this bound is even sharp, since G can be a disjoint union of stars $K_{1,\Delta}$, where A is taken as the set of all star centers and B as the set of all leaves. Likewise, we have

$$\begin{aligned} q_G(A,B) &\geq \frac{z(G \setminus (A \cup B))}{z(G \setminus B)} = \frac{z(G \setminus (A \cup B))}{\sum_{M \in \mathcal{M}_{G \setminus B}(A)} z(G \setminus (A \cup B \cup U(M)))} \\ &\geq |\mathcal{M}_{G \setminus B}(A)|^{-1} \geq |\mathcal{M}_G(A)|^{-1}. \end{aligned}$$

This readily proves the theorem in the case $d = 1$. For the induction step, we apply Lemma 5 again. Note that now $d > 1$, which means that vertices in A have no neighbors in B, thus implying $\mathcal{M}_{G \setminus B}(A) = \mathcal{M}_G(A)$. Hence we obtain

$$q_G(A,B) = \frac{z(G \setminus (A \cup B)) z(G)}{z(G \setminus A) z(G \setminus B)} = \frac{z(G \setminus (A \cup B)) \sum_{M \in \mathcal{M}_G(A)} z(G \setminus (A \cup U(M)))}{z(G \setminus A) \sum_{M \in \mathcal{M}_G(A)} z(G \setminus (A \cup B \cup U(M)))} . \tag{1}$$

If $U(M) \subseteq A$ (which happens in at least one case, namely $M = \emptyset$), we have

$$z(G \setminus (A \cup B \cup U(M))) = z(G \setminus (A \cup B)) \quad \text{and} \quad z(G \setminus (A \cup U(M))) = z(G \setminus A).$$

Otherwise, note that $|U(M) \setminus A| \leq |A| = a$ for every $M \in \mathcal{M}_G(A)$, since every edge of M has at least one end in A. Furthermore, every vertex in $U(M) \setminus A$ is a neighbor of A, so that $d(U(M), B) \geq d(A,B) - 1$ by the triangle inequality. Therefore, the induction hypothesis, applied to the new graph $H = G \setminus A$, shows that

$$\frac{1}{1 + C\,D^{d-2}} \leq q_H(U(M) \setminus A, B) = \frac{z(G \setminus (A \cup B \cup U(M))) z(G \setminus A)}{z(G \setminus (A \cup B)) z(G \setminus (A \cup U(M)))} \leq 1 + C\,D^{d-2}$$

for all $M \in \mathcal{M}_G(A)$. These observations, together with (1), yield

$$\begin{aligned} q_G(A,B) &= \frac{\sum_{M \in \mathcal{M}_G(A)} \frac{z(G \setminus (A \cup U(M)))}{z(G \setminus A)}}{\sum_{M \in \mathcal{M}_G(A)} \frac{z(G \setminus (A \cup B \cup U(M)))}{z(G \setminus (A \cup B))}} = \frac{1 + \sum_{M \in \mathcal{M}_G(A), M \neq \emptyset} \frac{z(G \setminus (A \cup U(M)))}{z(G \setminus A)}}{1 + \sum_{M \in \mathcal{M}_G(A), M \neq \emptyset} \frac{z(G \setminus (A \cup B \cup U(M)))}{z(G \setminus (A \cup B))}} \\ &\geq \frac{1 + \sum_{M \in \mathcal{M}_G(A), M \neq \emptyset} \frac{z(G \setminus (A \cup U(M)))}{z(G \setminus A)}}{1 + (1 + C\,D^{d-2}) \cdot \sum_{M \in \mathcal{M}_G(A), M \neq \emptyset} \frac{z(G \setminus (A \cup U(M)))}{z(G \setminus A)}} \\ &\geq \frac{|\mathcal{M}_G(A)|}{1 + (1 + C\,D^{d-2})(|\mathcal{M}_G(A)| - 1)} = \frac{1}{1 + CD^{d-2}(1 - |\mathcal{M}_G(A)|^{-1})} \end{aligned}$$

and analogously

$$q_G(A,B) \geq \frac{1}{1 + CD^{d-2}(1 - |\mathcal{M}_G(B)|^{-1})} .$$

Furthermore, the same argument shows that

$$\begin{aligned} q_G(A,B) &\leq \frac{|\mathcal{M}_G(A)|}{1 + (1 + C\,D^{d-2})^{-1}(|\mathcal{M}_G(A)| - 1)} \\ &= 1 + \frac{CD^{d-2}(|\mathcal{M}_G(A)| - 1)}{CD^{d-2} + |\mathcal{M}_G(A)|} \leq 1 + CD^{d-2}(1 - |\mathcal{M}_G(A)|^{-1}) \end{aligned}$$

and

$$q_G(A,B) \leq \frac{|\mathcal{M}_G(B)|}{1 + (1 + C\,D^{d-2})^{-1}(|\mathcal{M}_G(B)| - 1)} \leq 1 + CD^{d-2}(1 - |\mathcal{M}_G(B)|^{-1}).$$

Thus the induction is complete once we have $D \geq 1 - \min\{|\mathcal{M}_G(A)|, |\mathcal{M}_G(B)|\}^{-1}$, which holds for the given choice of $D = 1 - (1 + \Delta)^{-\min\{a,b\}}$, since

$$\min\{|\mathcal{M}_G(A)|, |\mathcal{M}_G(B)|\} \leq (1 + \Delta)^{\min\{a,b\}}$$

as before. ∎

4. Proofs of the Corollaries

This section is devoted to the proofs of the two corollaries that follow immediately from our main result. The first part is very easy: for a proof of Corollary 2, simply note that

$$\frac{r_G\big(\bigcup_{i=1}^k A_i\big)}{\prod_{i=1}^k r_G(A_i)} = \prod_{j=1}^{k-1} \frac{r_G\big(\bigcup_{i=1}^{j+1} A_i\big)}{r_G\big(\bigcup_{i=1}^{j} A_i\big)\, r_G(A_{j+1})} = \prod_{j=1}^{k-1} q_G\left(\bigcup_{i=1}^{j} A_i, A_{j+1}\right)$$

and that $d\big(\bigcup_{i=1}^j A_i, A_{j+1}\big) \geq \min_{r,s} d(A_r, A_s)$. ■

The proof of Corollary 3 is more intricate and involves an application of Lemma 6. Decompose A and B into sets V_A, V_B of vertices and sets E_A, E_B of edges. Without loss of generality, we may assume that no edge of E_A (E_B) has an endpoint in V_A (V_B, respectively). Now Lemma 6 yields

$$\begin{aligned} q_G(A,B) &= \frac{z(G \setminus (V_A \cup V_B \cup E_A \cup E_B))z(G)}{z(G \setminus (V_B \cup E_B))z(G \setminus (V_A \cup E_A))} \\ &= \frac{z(G \setminus (V_A \cup V_B \cup E_A \cup E_B))}{\sum_{M_1 \in \mathcal{I}(E_A)} z(G \setminus (V_B \cup E_B \cup E_A \cup U(M_1)))} \\ &\qquad \times \frac{\sum_{M_1 \in \mathcal{I}(E_A)} \sum_{M_2 \in \mathcal{I}(E_B)} z(G \setminus (E_A \cup E_B \cup U(M_1) \cup U(M_2)))}{\sum_{M_2 \in \mathcal{I}(E_B)} z(G \setminus (V_A \cup E_A \cup E_B \cup U(M_2)))}. \end{aligned}$$

Setting $H := G \setminus (E_A \cup E_B)$, this can be written as

$$q_G(A,B) = \frac{\sum_{M_1 \in \mathcal{I}(E_A)} \sum_{M_2 \in \mathcal{I}(E_B)} z(H \setminus (V_A \cup V_B))z(H \setminus (U(M_1) \cup U(M_2)))}{\sum_{M_1 \in \mathcal{I}(E_A)} \sum_{M_2 \in \mathcal{I}(E_B)} z(H \setminus (V_B \cup U(M_1)))z(H \setminus (V_A \cup U(M_2)))}.$$

Obviously, this can be estimated above by

$$\begin{aligned} &\max_{\substack{M_1 \in \mathcal{I}(E_A) \\ M_2 \in \mathcal{I}(E_B)}} \frac{z(H \setminus (V_A \cup V_B))z(H \setminus (U(M_1) \cup U(M_2)))}{z(H \setminus (V_B \cup U(M_1)))z(H \setminus (V_A \cup U(M_2)))} \\ &\qquad = \max_{\substack{M_1 \in \mathcal{I}(E_A) \\ M_2 \in \mathcal{I}(E_B)}} \frac{q_H(V_A, V_B) q_H(U(M_1), U(M_2))}{q_H(V_B, U(M_1)) q_H(V_A, U(M_2))} \leq (1 + C\, D^{d-1})^4 \end{aligned}$$

and below by $(1 + C\, D^{d-1})^{-4}$ in an analogous way, just as claimed. ■

5. An application to asymptotic enumeration

Chang and Chen [3] determined the asymptotics of the number of matchings in Sierpiński graphs, i.e. the finite approximations of the Sierpiński gasket. The motivation for these investigations lies in the fact that the enumeration of matchings is equivalent to the dimer-monomer model from statistical physics: dimers correspond to the matching edges, whereas monomers correspond to uncovered vertices. Their approach uses recurrence relations that can be deduced from the inductive construction of the finite Sierpiński graphs—the graph G_{n+1} is obtained by amalgamating three copies of G_n (see Figure 1).

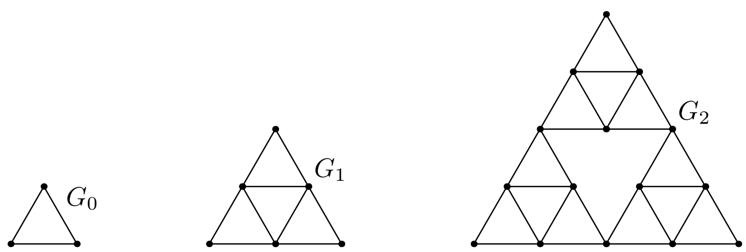

Figure 1. Steps in the construction of finite Sierpiński graphs.

A general setting for enumeration problems of this type has been provided in [16], including several examples of how the resulting recurrences can be treated. In the present case, one has to define a few auxiliary quantities: if $v_{1,n}, v_{2,n}, v_{3,n}$ are the corner vertices of G_n, we let

- a_n be the number of matchings of G_n with no edge incident to $v_{1,n}$, $v_{2,n}$ or $v_{3,n}$,
- b_n be the number of matchings of G_n with no edge incident to $v_{1,n}$ or $v_{2,n}$, but with an edge incident to $v_{3,n}$,
- c_n be the number of matchings of G_n with no edge incident to $v_{1,n}$, but with edges incident to $v_{2,n}$ and $v_{3,n}$,
- d_n be the number of matchings of G_n with edges incident to $v_{1,n}$, $v_{2,n}$ and $v_{3,n}$.

Obviously, b_n and c_n have three equivalent definitions for reasons of symmetry, and it follows that

$$z(G_n) = a_n + 3b_n + 3c_n + d_n.$$

All these quantities can be written in terms of $z(\cdot)$ by means of the inclusion-exclusion-principle, e.g.

$$c_n = z(G_n \setminus \{v_{1,n}\}) - z(G_n \setminus \{v_{1,n} \cup v_{2,n}\}) - z(G_n \setminus \{v_{1,n} \cup v_{3,n}\}) + z(G_n \setminus \{v_{1,n} \cup v_{2,n} \cup v_{3,n}\}).$$

The construction of G_{n+1} from G_n implies the following recurrence equations, which are given in [3]:

$$\begin{aligned}
a_{n+1} &= a_n^3 + 6a_n^2 b_n + 3a_n^2 c_n + 9a_n b_n^2 + 2b_n^3 + 6a_n b_n c_n,\\
b_{n+1} &= a_n^2 b_n + 2a_n^2 c_n + 4a_n b_n^2 + a_n^2 d_n + 8a_n b_n c_n + 3b_n^3 + 2a_n b_n d_n + 2a_n c_n^2 + 4b_n^2 c_n,\\
c_{n+1} &= a_n b_n^2 + 4a_n b_n c_n + 2b_n^3 + 2a_n b_n d_n + 7b_n^2 c_n + 3a_n c_n^2 + 2a_n c_n d_n + 2b_n^2 d_n + 4b_n c_n^2,\\
d_{n+1} &= b_n^3 + 6b_n^2 c_n + 3b_n^2 d_n + 9b_n c_n^2 + 2c_n^3 + 6b_n c_n d_n.
\end{aligned}$$

Now let us introduce three further quantities, namely

$$w_n = \frac{b_n}{a_n}, \quad x_n = \frac{c_n}{b_n}, \quad \text{and} \quad y_n = \frac{d_n}{c_n}.$$

A main step in the derivation of the asymptotics of a_n, b_n, c_n, d_n and finally $z(G_n)$ is to show that w_n, x_n, y_n tend to a common limit as $n \to \infty$. The proof given by Chang and Chen involves rather tedious calculations, which can be shortened considerably by means of our main theorem. Define R_n by

$$R_n = r_{G_n}(\{v_{i,n}\}) = \frac{z(G_n \setminus \{v_{i,n}\})}{z(G_n)}.$$

for $i \in \{1,2,3\}$ (note that the value does not depend on i). Then Corollary 2 shows that

$$z(G_n \setminus \{v_{i,n}, v_{j,n}\}) = z(G_n)(R_n^2 + O(D^{2^n}))$$

for some absolute positive constant $D < 1$ and $i, j \in \{1,2,3\}$, since the distance between $v_{i,n}$ and $v_{j,n}$ is 2^n. Similarly,

$$z(G_n \setminus \{v_{1,n}, v_{2,n}, v_{3,n}\}) = z(G_n)(R_n^3 + O(D^{2^n})).$$

It follows that

$$w_n = \frac{z(G_n)(R_n^2 - R_n^3 + O(D^{2^n}))}{z(G_n)(R_n^3 + O(D^{2^n}))} = R_n^{-1} - 1 + O(D^{2^n})$$

and similarly

$$x_n = R_n^{-1} - 1 + O(D^{2^n}) = w_n + O(D^{2^n}) \quad \text{and} \quad y_n = R_n^{-1} - 1 + O(D^{2^n}) = w_n + O(D^{2^n}).$$

It remains to show that these quantities tend to a limit. But plugging these formulæ for w_n, x_n, y_n into the recurrence equations yields

$$a_{n+1} = a_n^3(1 + 6w_n + 12w_n^2 + 8w_n^3 + O(D^{2^n}))$$

and

$$b_{n+1} = a_n^3(w_n + 6w_n^2 + 12w_n^3 + 8w_n^4 + O(D^{2^n})),$$

so that $w_{n+1} = w_n + O(D^{2^n})$, which shows that w_n (and thus x_n and y_n) converge to a limit W, and $w_n = W + O(D^{2^n})$. So finally

$$a_{n+1} = a_n^3(1 + 6W + 12W^2 + 8W^3 + O(D^{2^n})),$$

and taking logarithms yields

$$\log a_{n+1} = 3 \log a_n + K + O(D^{2^n})$$

for some constant $K = \log(1 + 6W + 12W^2 + 8W^3)$. Writing c_n for the error term and iterating this recursion (cf. [1]), we find

$$\begin{aligned}\log a_n &= 3^n \log a_0 + \frac{3^n - 1}{2} K + \sum_{k=0}^{n-1} 3^{n-k-1} c_k \\ &= 3^n \left(\log a_0 + \frac{K}{2} + \sum_{k=0}^{\infty} 3^{-k-1} c_k \right) - \frac{K}{2} - \sum_{k=n}^{\infty} 3^{n-k-1} c_k \\ &= A \cdot 3^n + B + O(D^{2^n})\end{aligned}$$

for constants A and $B = -K/2$, and therefore

$$a_n = \beta \cdot \alpha^{3^n} \left(1 + O(D^{2^n})\right)$$

with $\alpha = e^A$ and $\beta = e^B$. The quantities b_n, c_n and d_n satisfy analogous asymptotic equations, and since $z(G_n) = a_n + 3b_n + 3c_n + d_n$, we can conclude with the following theorem:

Theorem 7. *The number of matchings in the finite Sierpiński graph G_n of level n is (asymptotically)*

$$z(G_n) = \gamma \cdot \alpha^{3^n} \left(1 + O(D^{2^n})\right)$$

for positive constants α and γ.

Numerical values for α and γ are given by

$$\alpha = 2.676316356977$$

and

$$\gamma = 1.427712384869.$$

The value of the "growth constant"

$$\lim_{n\to\infty} \frac{\log z(G_n)}{|V(G_n)|} = \tfrac{2}{3} \log \alpha = 0.2850249655977$$

has already been provided by Chang and Chen in [3]. Of course, our method is not restricted to this special case and can be used for generalized Sierpiński graphs in higher dimensions and with more subdivisions. Some examples for this generalization have already been discussed by Chang and Chen as well.

6. Further remarks

Lots of open questions remain. For instance, it is natural to ask whether similar estimates can be given for other graph parameters (in special cases, we have already observed a similar behavior for the number of independent vertex subsets and for the number of maximal matchings in [16]). Since Corollary 2 and thus also Theorem 1 play the key role in the asymptotic enumeration problem considered in Section 5, we expect a similar benefit for problems involving other graph parameters. Thus we believe it might be worthwhile to study our independence phenomenon from a more general point of view.

Furthermore, the constants provided in Theorem 1 are most probably not best-possible, and there should be a lot of space for improvement, bearing in mind that the estimates in the proofs are rather crude. It is also not clear whether the dependence on the maximum degree of G is actually necessary, and it might be possible to drop it.

We also conjecture the following bounds for $q_G(A, B)$ that neither depend on Δ nor on the distance $d(A, B)$:

Conjecture 1. *For an arbitrary graph G and vertex subsets A, B of G with $a = |A|$ and $b = |B|$, we have*

$$\frac{z(K_{c,a+b})}{z(K_{c,a})z(K_{c,b})} \leq q_G(A, B) \leq z(K_{a,b}),$$

where $c = \lfloor \frac{a+b+2}{2} \rfloor$, unless $a = b = 1$ or $a = b = 2$. In these two cases, $c = 1$ and $c = 2$ respectively. Equality occurs for the left hand side inequality if and only if G is the complete bipartite graph $K_{c,a+b}$, with bipartition $(C, A \cup B)$, and on the right hand side if and only if G is the complete bipartite graph $K_{a,b}$ with bipartition (A, B).

Finally, let us mention that there are interesting instances where the quotient $q_{G_n}(A_n, B_n)$ tends to 1 for certain sequences $(G_n)_{n\geq 1}$, $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$ even though the distance is bounded, and we will exhibit such an example below. On the other hand, this condition cannot be dropped, as the following simple example shows:

Let $G_n = P_{2n}$ be a path on $2n$ vertices, and let v_n, w_n be the two middle vertices, i.e. those with largest distance from the leaves. It is well-known and easy to prove that $z(P_m) = F_{m+1}$ is a Fibonacci number. Hence we have

$$\begin{aligned} z_{G_n}(\{v_n\}, \{w_n\}) &= \frac{z(G_n \setminus \{v_n, w_n\})z(G_n)}{z(G_n \setminus \{v_n\})z(G_n \setminus \{w_n\})} \\ &= \frac{z(P_{n-1})^2 z(P_{2n})}{(z(P_n)z(P_{n-1}))^2} = \frac{F_n^2 F_{2n+1}}{F_{n+1}^2 F_n^2} = \frac{F_{2n+1}}{F_{n+1}^2}, \end{aligned}$$

which tends to $\frac{5-\sqrt{5}}{2} \neq 1$. On the other hand, if we take $G_n = K_n$, then we obtain

$$q_{G_n}(\{v_n\}, \{w_n\}) = \frac{z(K_n)z(K_{n-2})}{z(K_{n-1})^2}$$

for arbitrary distinct vertices $v_n, w_n \in G_n$. The asymptotics of $z(K_n)$ have been given in [4], albeit in a different context ($z(K_n)$ is also, among others, the number of involutions, i.e. permutations of order 2; see [15], A000085): one has

$$z(K_n) \sim \exp\left(\frac{n \log n}{2} - \frac{n}{2} + \sqrt{n} - \frac{1}{4} - \frac{\log 2}{2}\right),$$

which shows that

$$q_{G_n}(\{v_n\}, \{w_n\}) \to 1$$

as $n \to \infty$.

Finally, let us mention a question stemming from the observation that finite Sierpiński graphs can be seen as induced subgraphs of an infinite graph: under which circumstances is it true that

$$\lim_{n\to\infty} \frac{\log z(G_n)}{|V(G_n)|},$$

where G_n is an (appropriate) increasing sequence of induced finite subgraphs of a locally finite graph G, is independent of the sequence G_n (i.e., depends only on G)?

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