THE DISTRIBUTION OF ASCENTS OF SIZE d OR MORE IN PARTITIONS OF n

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ABSTRACT. A partition of a positive integer n is a finite sequence of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1+a_2+\cdots+a_k = n$ and $a_{i+1} \geq a_i$ for all i. Let d be a fixed positive integer. We say that we have an ascent of size d or more if $a_{i+1} \geq a_i + d$. We determine the mean, the variance and the limiting distribution of the number of ascents of size d or more in the partitions of n.

1. INTRODUCTION

A partition of a positive integer n is a finite sequence of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \cdots + a_k = n$ and $a_{i+1} \geq a_i$ for all i. We say n is the size of the partition, a_i is the ith part of the partition and we call $p(n)$ the number of partitions of n.
For instance the 11 partitions of $n = 6$ are 6, 15, 24, 33, 222, 123, 114, 1113, 1122, 11112 and 111111, i.e., $p(6) = 11$.
We define an ascent of size d or more whenever $a_{i+1} \geq a_i + d$. In this paper we aim to look at the distribution of the number of ascents of size d or more in the partitions of n. The case for $d = 0$, equivalent to the number of parts in partitions of n, was first studied by P. Erdős and J. Lehner in [5]. Henceforth we will restrict our attention to the case where $d \geq 1$.

In Section 2, we find an expression for the mean number of ascents of size d or more in the partitions of n. For this, we use a generating function and Ferrer's diagrams. If α_n is the number of ascents of size d or more in a random partition of n, we find that

$$\mathbb{E}(\alpha_n) = \frac{\sum\limits_{m\geq 1} p(n-md)}{p(n)}.$$

In Section 3, we proceed to find the variance $\mathbb{V}(\alpha_n)$, where for $d \geq 1$

$$\mathbb{V}(\alpha_n) = \frac{1}{p(n)}\sum_{i\geq 0}(i+1)p(n-d(3+i)) + \frac{1}{p(n)}\sum_{i\geq 0} p(n-d(3+2i)) + \mathbb{E}(\alpha_n) - \mathbb{E}(\alpha_n)^2.$$

In Section 4, we find an asymptotic expression for the mean and prove that as $n \to \infty$

$$\mathbb{E}(\alpha_n) \sim \frac{\sqrt{6n}}{\pi\, d} + \frac{3}{\pi^2\, d} - \frac{1}{2}.$$

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In Section 5, we find the following asymptotic expression for the variance:

$$\mathbb{V}(\alpha_n) \sim \frac{\sqrt{6n}\,(d\pi^2-6)}{2d^2\pi^3} + \frac{3}{2d\pi^2} - \frac{18}{d^2\pi^4}.$$

Finally, in Section 6, we show using the saddle point method that asymptotically this number of ascents follows a normal distribution with mean and variance found in Sections 4 and 5 respectively.

2. Generating function: Ferrers' diagrams

In this section, we find a connection between partitions with an ascent of size d and partitions with a part of multiplicity d. To show this, we first need to consider the Ferrers graphical representation of a partition, which is a collection of lattice points where each row of dots corresponds to a part of the partition, as seen in [1].
For instance, the Ferrers graphical representation of the partition of 13, 1+2+5+5 is

$$\begin{array}{cccccc}
1 & \bullet & & & & \\
2 & \bullet & \bullet & & & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & 4 & 3 & 2 & 2 & 2
\end{array}$$

If we add up the number of dots in each column, we obtain another partition called the conjugate of the original partition. In our example, the conjugate of 1+2+5+5 is 4+3+2+2+2.
The partition 1+2+5+5 has an ascent of size 3 between the parts 2 and 5. This, in turn, is reflected in the conjugate with a part of multiplicity 3. The part 2 is repeated 3 times.
The idea can be generalised, i.e., the conjugate of a partition with an ascent of size d or more has a part with multiplicity of at least d. Let us look at the generating function where z marks the size of the partition and u marks the parts with multiplicity of at least d.

$$\begin{aligned}
G_d(z,u) &= \prod_{i\geq 1}\left(1+z^i+z^{2i}+\cdots+z^{(d-1)i}+u\big(z^{di}+z^{(d+1)i}+\cdots\big)\right) \\
&= \prod_{i\geq 1}\left(\frac{1-z^{di}}{1-z^i}+u\frac{z^{di}}{1-z^i}\right) \qquad (2.1)\\
&= P(z)\prod_{i\geq 1}\left(1+(u-1)z^{di}\right)\,, \qquad (2.2)
\end{aligned}$$

where $P(z) = \prod\limits_{j\geq 1}\frac{1}{1-z^j} = \sum\limits_{n\geq 0} p(n)z^n$.

As usual, to obtain the number of parts with multiplicity of at least d, we differentiate with respect to u and put $u = 1$, to obtain

$$\frac{\partial}{\partial u} G_d(z,u)\Big|_{u=1} = P(z) \sum_{i\geq 1} z^{di} = P(z)\frac{z^d}{1-z^d}. \tag{2.3}$$

However, this is perhaps not totally correct! Let's look at, for instance, the partition of 9, 4+5.

$$\begin{array}{cccccc} 4 & \bullet & \bullet & \bullet & \bullet & \\ 5 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 2 & 2 & 2 & 2 & 1 \end{array}$$

It has no ascent of size $d = 3$ or more, but its conjugate does have a part 2, of multiplicity of 4 which is greater than 3. The above formula fails when the partition has a first part of size greater or equal to d, here the first part $4 \geq 3 = d$. For convenience, we will count a first element that is $\geq d$ as an ascent as well (one could argue that this is an ascent from 0), which simplifies the calculations. Asymptotically, this does not make a difference, except for the case $d = 1$; but even in this case the difference is marginal. Hence the final number of ascents is

$$[z^n]\left(P(z)\frac{z^d}{1-z^d}\right) = [z^{n-d}]\frac{1}{1-z^d}\sum_{n\geq 0} p(n)z^n \tag{2.4}$$

$$= \sum_{m\geq 1} p(n-md). \tag{2.5}$$

The expected number $\mathbb{E}(\alpha_n)$ of ascents per partition of n of size d or more is the number of ascents obtained in (2.5) divided by the total number $p(n)$ of partitions of n, hence:

Theorem 1. *The expected number of ascents per partition of n of size d or more is*

$$\mathbb{E}(\alpha_n) = \frac{\sum_{m\geq 1} p(n-md)}{p(n)} \quad \textit{for } d \geq 1. \tag{2.6}$$

Wilf in [9] studied the number of distinct part sizes in partitions, this corresponds to ascents of size d or more when $d = 1$. Knopfmacher and Warlimont in [7] studied details of gaps in partitions which is equivalent to ascents of size 2 or more. Also Corteel *et al.* in [3] have investigated various questions relating to distinct part sizes and multiplicity of parts in partitions.

3. Variance of the number of ascents of size d or more in the partitions of n

Again we shall use the generating function found in Section 2. Differentiating

$$G_d(z,u) = \prod_{i\geq 1}\left(\frac{1-z^{id}}{1-z^i} + u\frac{z^{id}}{1-z^i}\right) \tag{3.1}$$

twice with respect to u and setting $u=1$, we obtain

$$
\begin{aligned}
&\frac{\partial^2}{\partial u^2} G_d(z,u)\Big|_{u=1}\\
&= \prod_{i\geq 1} \frac{1}{1-z^i} \left(z^d \frac{z^d}{1-z^d} - z^{2d} + z^{2d}\frac{z^d}{1-z^d} - z^{4d} + z^{3d}\frac{z^d}{1-z^d} - z^{6d} + \cdots \right)\\
&= \sum_{i\geq 0} p(i) z^i \left(\left(\frac{z^d}{1-z^d}\right)^2 - \frac{z^{2d}}{1-z^{2d}} \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
[z^n]\frac{\partial^2}{\partial u^2} G_d(z,u)\Big|_{u=1} &= [z^n] \sum_{i\geq 0} p(i) z^i \; \frac{2z^{3d}}{(1-z^d)(1-z^{2d})}\\
&= [z^{n-3d}] \sum_{i\geq 0} p(i) z^i \left(\frac{1}{(1-z^d)^2} + \frac{1}{1-z^{2d}} \right)\\
&= [z^{n-3d}] \sum_{j\geq 0} p(j) z^j \left(\sum_{i\geq 0} (i+1) z^{id} + \sum_{i\geq 0} z^{2id} \right)\\
&= \sum_{i\geq 0} (i+1) p(n-d(3+i)) + \sum_{i\geq 0} p(n-d(3+2i)). \qquad (3.2)
\end{aligned}
$$

It follows that the variance is

$$
\begin{aligned}
\mathbb{V}(\alpha_n) &= \frac{1}{p(n)} \left([z^n]\frac{\partial^2}{\partial u^2} G_d(z,u)\Big|_{u=1} + [z^n]\frac{\partial}{\partial u} G_d(z,u)\Big|_{u=1} \right) - \mathbb{E}(\alpha_n)^2\\
&= \frac{1}{p(n)} \sum_{i\geq 0} (i+1) p(n-d(3+i)) + \frac{1}{p(n)} \sum_{i\geq 0} p(n-d(3+2i)) + \mathbb{E}(\alpha_n) - \mathbb{E}(\alpha_n)^2,
\end{aligned}
$$

so we have the following theorem:

Theorem 2. *The variance of the number of ascents per partition of n of size d or more is for $d \geq 1$*

$$
\mathbb{V}(\alpha_n) = \frac{1}{p(n)} \sum_{i\geq 0} (i+1) p(n-d(3+i)) + \frac{1}{p(n)} \sum_{i\geq 0} p(n-d(3+2i)) + \mathbb{E}(\alpha_n) - \mathbb{E}(\alpha_n)^2.
$$

4. Asymptotic expressions for the mean

It is of interest to see how the expected value $\mathbb{E}(\alpha_n)$ grows asymptotically as $n \to \infty$, where

$$
\mathbb{E}(\alpha_n) = \frac{\sum_{m\geq 1} p(n-md)}{p(n)} \quad \text{for} \quad d \geq 1\,.
$$

Let $n \equiv r (\text{mod } d)$ for $d \geq 2$ and set $r=0$ if $d=1$. Then

$$
\sum_{m\geq 1} p(n-md) = \sum_{m=0}^{\frac{n-d-r}{d}} p(md+r).
$$

We use the asymptotic estimate for $p(n)$

$$p(n) = e^{\pi\sqrt{\frac{2n}{3}}}\left(\frac{1}{4\sqrt{3}n} - \frac{72+\pi^2}{288\sqrt{2}\pi n^{3/2}} + O(n^{-2})\right), \tag{4.1}$$

which can be found by expanding the dominant term of Rademacher's series representation (see [2] for instance) for $p(n)$.

By means of the Euler-MacLaurin summation formula we have the following result

$$\sum_{j=1}^{n}\frac{e^{\alpha\sqrt{j}}}{j^{\beta}} = e^{\alpha\sqrt{n}}\left[\frac{2}{\alpha n^{\beta-\frac{1}{2}}} + \left(\frac{2(2\beta-1)}{\alpha^2}+\frac{1}{2}\right)\frac{1}{n^{\beta}} + O(n^{-\beta-1/2})\right]. \tag{4.2}$$

Thus, working up to order $O(\frac{1}{\sqrt{n}})$,

$$\begin{aligned}
p(md+r) &\sim \frac{1}{4\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2(md+r)}{3}}}}{md+r} - \frac{1}{\sqrt{2}\,\pi}\frac{e^{\pi\sqrt{\frac{2(md+r)}{3}}}(72+\pi^2)}{288(md+r)^{3/2}} \qquad \text{using (4.1)}\\
&= \frac{1}{4\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2md}{3}}(1+\frac{r}{md})^{1/2}}}{md(1+\frac{r}{md})} - \frac{72+\pi^2}{288\sqrt{2}\,\pi}\frac{e^{\pi\sqrt{\frac{2md}{3}}(1+\frac{r}{md})^{1/2}}}{(md)^{3/2}(1+\frac{r}{md})^{3/2}}\\
&\sim \frac{1}{4\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2md}{3}}\,(1+\frac{r}{2md})}}{md(1+\frac{r}{md})} - \frac{72+\pi^2}{288\sqrt{2}\,\pi}\frac{e^{\pi\sqrt{\frac{2md}{3}}\,(1+\frac{r}{2md})}}{(md)^{3/2}(1+\frac{r}{md})^{3/2}}\\
&\sim \frac{1}{4\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{md}\left(1+\frac{\pi r}{\sqrt{6md}}\right)\left(1-\frac{r}{md}\right) - \frac{72+\pi^2}{288\sqrt{2}\,\pi}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{(md)^{3/2}}\left(1+\frac{\pi r}{\sqrt{6md}}\right)\left(1-\frac{3r}{2md}\right).
\end{aligned}$$

Thus

$$p(md+r) \sim \frac{1}{4\sqrt{3}}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{md}\left(1+\frac{\pi r}{\sqrt{6md}}\right) - \frac{72+\pi^2}{288\sqrt{2}\,\pi}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{(md)^{3/2}}.$$

We now sum over m from $m=1$ as the $m=0$ term is insignificant. For the accuracy that is required we shall only take the terms that have $\beta = 1$ or $3/2$ and $\alpha = \pi\sqrt{\frac{2d}{3}}$ in (4.2), the expansion of $\sum\limits_{m=1}^{n}\frac{e^{\alpha\sqrt{m}}}{m^{\beta}}$ and obtain

$$\begin{aligned}
&\sum_{m=1}^{\frac{n-d-r}{d}} p(md+r)\\
&\sim \frac{1}{4\sqrt{3}\,d}\sum_{m=1}^{\frac{n-d-r}{d}}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{m} + \frac{\pi\, r}{12\sqrt{2}\,d^{3/2}}\sum_{m=1}^{\frac{n-d-r}{d}}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{m^{3/2}} - \frac{72+\pi^2}{288\sqrt{2}\,\pi\, d^{3/2}}\sum_{m=1}^{\frac{n-d-r}{d}}\frac{e^{\pi\sqrt{\frac{2md}{3}}}}{m^{3/2}}\\
&\sim \frac{1}{4\sqrt{3}}e^{\pi\sqrt{\frac{2n}{3}}}e^{-\pi\frac{d+r}{\sqrt{6n}}}\left[\frac{\sqrt{6}\,(1+\frac{d+r}{2n})}{\pi d\sqrt{n}} + \left(\frac{3}{\pi^2 d}+\frac{1}{2}\right)\frac{1+\frac{d+r}{n}}{n}\right]\\
&+ \frac{\pi\, r}{12\sqrt{2}}e^{\pi\sqrt{\frac{2n}{3}}}e^{-\pi\frac{d+r}{\sqrt{6n}}}\left[\frac{\sqrt{6}\,(1+\frac{d+r}{n})}{\pi d\, n} + \left(\frac{6}{\pi^2 d}+\frac{1}{2}\right)\frac{1+\frac{3(d+r)}{2n}}{n^{3/2}}\right]\\
&- \frac{72+\pi^2}{288\sqrt{2}\,\pi}e^{\pi\sqrt{\frac{2n}{3}}}e^{-\pi\frac{d+r}{\sqrt{6n}}}\left[\frac{\sqrt{6}\,(1+\frac{d+r}{n})}{\pi d\, n} + \left(\frac{6}{\pi^2 d}+\frac{1}{2}\right)\frac{1+\frac{3(d+r)}{2n}}{n^{3/2}}\right].
\end{aligned} \tag{4.3}$$

We are ready to divide by $p(n)$ using

$$\frac{1}{p(n)} \sim \frac{4\sqrt{3}\,n}{e^{\pi\sqrt{\frac{2n}{3}}}}\left[1+\frac{(72+\pi^2)\sqrt{3}}{72\sqrt{2n}\,\pi}\right]. \tag{4.4}$$

We also estimate

$$e^{-\pi\frac{d+r}{\sqrt{6n}}} \quad \text{by} \quad 1-\frac{\pi(d+r)}{\sqrt{6n}}. \tag{4.5}$$

Thus, keeping only the terms involving $\sqrt{n}$ and constants we obtain after multiplying (4.3) by (4.4) and substituting (4.5)

$$\begin{aligned}\frac{\sum_{m=1}^{\frac{n-d-r}{d}} p(md+r)}{p(n)} &\sim \frac{\sqrt{6n}}{\pi d}+\frac{72+\pi^2}{24\,\pi^2 d}+\frac{3}{\pi^2\,d}+\frac{1}{2}-\frac{d+r}{d}+\frac{r}{d}-\frac{72+\pi^2}{24\,\pi^2\,d}\\ &\sim \frac{\sqrt{6n}}{\pi\,d}+\frac{3}{\pi^2\,d}-\frac{1}{2}.\end{aligned}$$

So, finally

Theorem 3. *The expected number of ascents of size d or more in the partitions of n is*

$$\mathbb{E}(\alpha_n) \sim \frac{\sqrt{6n}}{\pi\,d}+\frac{3}{\pi^2\,d}-\frac{1}{2} \quad \textit{as } n\to\infty.$$

5. Asymptotic expressions for the variance

Theorem 4. *For $d \geq 1$ the variance for the number of ascents of size d or more satisfies*

$$\mathbb{V}(\alpha_n) \sim \frac{\sqrt{6n}\,(d\pi^2-6)}{2d^2\pi^3}+\frac{3}{2d\pi^2}-\frac{18}{d^2\pi^4}.$$

Proof. Since the sums in the variance include terms of magnitude $np(n)$ we require more precise asymptotic estimates than in the case of the mean in order to correctly compute the constant term in the variance. For the number of partitions $p(n)$ we use the asymptotic estimate

$$p(n) = e^{\pi\sqrt{\frac{2n}{3}}}\left(\frac{1}{4\sqrt{3}n}-\frac{72+\pi^2}{288\sqrt{2}\pi n^{3/2}}+\frac{432+\pi^2}{27648\sqrt{3}n^2}+O(n^{-5/2})\right) \tag{5.1}$$

which can be found by further expanding the dominant term of Rademacher's series representation for $p(n)$.

Via the Euler-Maclaurin summation formula we get the more precise asymptotic formula for $\sum_{1\le k\le n}\frac{e^{\sqrt{k}\alpha}}{k^\beta}$

$$\begin{aligned}\sum_{1\le k\le n}\frac{e^{\sqrt{k}\alpha}}{k^\beta} &\sim e^{\sqrt{n}\alpha}\left(\frac{2}{\alpha n^{\beta-\frac{1}{2}}}+\frac{\alpha^2+8\beta-4}{2\alpha^2\,n^\beta}+\frac{\alpha^4+96\beta(2\beta-1)}{24\alpha^3\,n^{\beta+\frac{1}{2}}}\right)\\ &+e^{\sqrt{n}\alpha}\left(-\frac{\beta\,(\alpha^4-192\beta^2+48)}{12\alpha^4\,n^{\beta+1}}-\frac{\alpha^8-46080\beta\,(4\beta^3+4\beta^2-\beta-1)}{5760\alpha^5\,n^{\beta+\frac{3}{2}}}\right).\end{aligned} \tag{5.2}$$

For the first sum in the variance

$$s_1 := \frac{1}{p(n)} \sum_{i \geq 0} p(n - d(3+2i))$$

we can use the same asymptotic estimates as used to find the mean to obtain

$$s_1 \sim \frac{\sqrt{\frac{3}{2}}\sqrt{n}}{d\pi} + \frac{3}{2d\pi^2} - 1.$$

For the other sum in the variance the more precise asymptotic estimates (5.1) and (5.2) are required. For

$$s_2 := \frac{1}{p(n)} \sum_{i \geq 0} (i+1) p(n - d(3+i)) = \frac{1}{p(n)} \sum_{m=0}^{\frac{n-3d-r}{d}} \left(\frac{-2d+n-r}{d} - m \right) p(dm+r)$$

where $n - d(3+i) \equiv r \pmod d$ with $0 \leq r \leq d-1$, we eventually obtain

$$s_2 \sim \frac{6n}{d^2\pi^2} + \frac{\sqrt{6}\,(3 - 2d\pi^2)\sqrt{n}}{d^2\pi^3} + \frac{11\pi^4 d^2 - 3(24d+1)\pi^2 + 108}{12 d^2 \pi^4} + 1. \tag{5.3}$$

We must also compute a more precise estimate for the mean value

$$\mathbb{E}(\alpha_n) = \frac{\sqrt{6}\sqrt{n}}{d\pi} + \frac{3}{d\pi^2} - \frac{1}{2} + \frac{2\pi^4 d^2 - 3\pi^2 + 216}{24\sqrt{6} d \sqrt{n} \pi^3}.$$

The $1/\sqrt{n}$ term above is needed in order to find the constant term in $\mathbb{E}(\alpha_n)^2$. This gives for the variance the estimate

$$\mathbb{V}(\alpha_n) = s_1 + s_2 + \mathbb{E}(\alpha_n) - \mathbb{E}(\alpha_n)^2 \sim \frac{\sqrt{6n}\,(d\pi^2 - 6)}{2d^2\pi^3} + \frac{3}{2d\pi^2} - \frac{18}{d^2\pi^4}$$

as claimed.

6. Limiting distribution

In this section, we are going to show that the number of ascents of size at least d in a random number partition asymptotically follows a normal law. The proof will run along the same lines as in Hwang's paper [6]. However, since our setting is less general than Hwang's, some of his methods can be replaced by simpler ones. In order to make this paper self-contained, we are going to give almost all details. Again, we start with the generating function

$$G_d(z) := G_d(z,u) = \prod_{j=1}^{\infty} \frac{1 + (u-1)z^{dj}}{1 - z^j}.$$

Then we know that

$$Q_n(u) := [z^n] G_d(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} z_0^{-n} e^{-int} G_d(z_0 e^{it}, u)\, dt$$

for an arbitrary positive real number $z_0 < 1$. We are going to consider the integrand's logarithm

$$g_d(z) = \sum_{j=1}^{\infty} \log(1 + (u-1)z^{dj}) - \sum_{j=1}^{\infty} \log(1 - z^j) - n \log z.$$

In order to apply the saddle point method, we have to determine z_0 in such a way that

$$g'(z_0) = \sum_{j=1}^{\infty} \frac{dj(u-1)z_0^{dj-1}}{1+(u-1)z_0^{dj}} + \sum_{j=1}^{\infty} \frac{jz_0^{j-1}}{1-z_0^j} - \frac{n}{z_0} = 0. \tag{6.1}$$

We write $e^{-\beta}$ for z_0, which yields the equation

$$\sum_{j=1}^{\infty} \frac{dj(u-1)e^{-dj\beta}}{1+(u-1)e^{-dj\beta}} + \sum_{j=1}^{\infty} \frac{je^{-j\beta}}{1-e^{-j\beta}} = \sum_{j=1}^{\infty} \frac{dj(u-1)}{u-1+e^{dj\beta}} + \sum_{j=1}^{\infty} \frac{j}{e^{j\beta}-1} = n.$$

Next, we want to apply the Euler-Maclaurin summation formula. To this end, we use the following integral representation of the second-order polylogarithm $\mathrm{Li}_2(x)$ (which can be written as $\sum_{k\geq 1} \frac{x^k}{k^2}$ for $|x| < 1$, see [8]):

Lemma 1.

$$\int_0^{\infty} \frac{t\,dt}{v+e^t} = -\frac{\mathrm{Li}_2(-v)}{v}$$

for $v \geq -1$. In the special cases $v = -1$ and $v = 0$, we obtain $\frac{\pi^2}{6}$ and 1 respectively.

In the following, we write v for $u-1$. Furthermore, we fix a real number $\delta > 0$ and assume $\delta \leq u \leq \delta^{-1}$. Thus, "uniformly" is supposed to mean "uniformly for $\delta \leq u \leq \delta^{-1}$" or "uniformly for $\delta - 1 \leq v \leq \delta^{-1} - 1$". We have, by the Euler-Maclaurin summation formula,

$$\sum_{j=1}^{\infty} \frac{j}{e^{j\beta}-1} = \frac{\pi^2}{6\beta^2} - \frac{1}{2\beta} + O(1),$$

and

$$\sum_{j=1}^{\infty} \frac{djv}{v+e^{dj\beta}} = -\frac{\mathrm{Li}_2(-v)}{d\beta^2} + O_\delta(1),$$

the latter holding uniformly in v. Hence, we can write equation (6.1) as

$$\left(\frac{\pi^2}{6} - \frac{\mathrm{Li}_2(1-u)}{d}\right)\beta^{-2} - \frac{1}{2\beta} + O_\delta(1) = n,$$

which means that the solution to this equation is $z_0 = e^{-\beta}$, where

$$\beta = \frac{b}{\sqrt{n}} - \frac{1}{4n} + O_\delta(n^{-3/2}).$$

$b = b(u)$ is used as an abbreviation for $\sqrt{\frac{\pi^2}{6} - \frac{\mathrm{Li}_2(1-u)}{d}}$ in this and the subsequent formulas.

In a similar manner, we have

$$-\sum_{j=1}^{\infty} \log(1-e^{-j\beta}) = \frac{\pi^2}{6\beta} + \frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) + O(\beta)$$

and

$$\sum_{j=1}^{\infty} \frac{j^2 e^{-j\beta}}{(1-e^{-j\beta})^2} = \frac{\pi^2}{3\beta^3} - \frac{1}{2\beta^2} + O(\beta)$$

as well as

$$\sum_{j=1}^{\infty} \log(1+ve^{-dj\beta}) = -\frac{\mathrm{Li}_2(-v)}{d\beta} - \frac{1}{2}\log(1+v) + O_\delta(\beta)$$

and

$$\sum_{j=1}^{\infty} \frac{j^2 e^{-dj\beta}}{(1+ve^{-dj\beta})^2} = -\frac{2\,\mathrm{Li}_2(-v)}{d^3 v\beta^3} + O_\delta(\beta),$$

uniformly in v. Thus we have

$$\begin{aligned} g_d(e^{-\beta}) &= \sum_{j=1}^{\infty} \log(1+(u-1)e^{-dj\beta}) - \sum_{j=1}^{\infty} \log(1-e^{-j\beta}) + n\beta \\ &= \left(\frac{\pi^2}{6} - \frac{\mathrm{Li}_2(1-u)}{d}\right)\beta^{-1} + \frac{1}{2}\log\left(\frac{\beta}{2\pi u}\right) + n\beta + O_\delta(\beta) \\ &= 2b\sqrt{n} + \frac{1}{2}\log\left(\frac{b}{2\pi u\sqrt{n}}\right) + O_\delta(n^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} g''(e^{-\beta}) &= \sum_{j=1}^{\infty}\left(\frac{d^2 j^2 (u-1)e^{-(dj-2)\beta}}{(1+(u-1)e^{-dj\beta})^2} - \frac{dj(u-1)e^{-(dj-2)\beta}}{1+(u-1)e^{-dj\beta}} + \frac{j^2 e^{-(j-2)\beta}}{(1-e^{-j\beta})^2} - \frac{je^{-(j-2)\beta}}{1-e^{-j\beta}}\right) + ne^{2\beta} \\ &= e^{2\beta}\left(n + \sum_{j=1}^{\infty} \frac{d^2 j^2 (u-1)e^{-dj\beta}}{(1+(u-1)e^{-dj\beta})^2} - \frac{dj(u-1)e^{-dj\beta}}{1+(u-1)e^{-dj\beta}} + \frac{j^2 e^{-j\beta}}{(1-e^{-j\beta})^2} - \frac{je^{-j\beta}}{1-e^{-j\beta}}\right) \\ &= \frac{2b^2}{\beta^3} + O_\delta(\beta^{-2}) = \frac{2n^{3/2}}{b} + O_\delta(n). \end{aligned}$$

It is also not difficult to show that $g'''(e^{-\beta}) = O_\delta(n^2)$. All the estimates hold uniformly in u.
Now we need a uniform estimate of $G_d(z)$ when z is away from the saddle point. We set $z = z_0 e^{it}$ and use the abbreviation $w = 1-u$. Then we have

$$\begin{aligned} \left|\frac{G_d(z_0)}{G_d(z)}\right|^2 &= \prod_{j\geq 1}\left|\frac{1-wz_0^{dj}}{1-wz^{dj}}\right|^2 \prod_{j\geq 1}\left|\frac{1-z^j}{1-z_0^j}\right|^2 \\ &\geq \prod_{j\geq 1}\min\left(1, \left|\frac{1-wz_0^{dj}}{1-wz^{dj}}\right|^2\right) \prod_{j\geq 1}\left|\frac{1-z^j}{1-z_0^j}\right|^2 \\ &\geq \prod_{j\geq 1}\min\left(1, \left|\frac{1-wz_0^{j}}{1-wz^{j}}\right|^2\right) \left|\frac{1-z^j}{1-z_0^j}\right|^2 \\ &= \prod_{j\geq 1}\min\left(1, \frac{(1-wz_0^j)^2}{(1-wz_0^j)^2 + 2wz_0^j(1-\cos(tj))}\right) \frac{(1-z_0^j)^2 + 2z_0^j(1-\cos(tj))}{(1-z_0^j)^2} \\ &= \prod_{j\geq 1}\left(1 + \max\left(0, \frac{2wz_0^j(1-\cos(tj))}{(1-wz_0^j)^2}\right)\right)^{-1}\left(1 + \frac{2z_0^j(1-\cos(tj))}{(1-z_0^j)^2}\right) \end{aligned}$$

$$\geq \prod_{j\geq 1}\left(1+\max\left(0,\ \frac{2wz_0^j(1-\cos(tj))}{(1-z_0^j)^2}\right)\right)^{-1}\left(1+\frac{2z_0^j(1-\cos(tj))}{(1-z_0^j)^2}\right).$$

$$\geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+\max\left(0,\ \frac{2wz_0^j(1-\cos(tj))}{(1-z_0^j)^2}\right)\right)^{-1}\left(1+\frac{2z_0^j(1-\cos(tj))}{(1-z_0^j)^2}\right).$$

Since $z_0^{\sqrt{n}} \to e^{-b}$, we know that z_0^j is bounded below for $\sqrt{n}\leq j\leq 2\sqrt{n}$. This bound holds uniformly in u again. Consequently, the same is true for $\frac{2z_0^j}{(1-z_0^j)^2}$. Let $A = A(\delta)$ be some lower bound for $\frac{2z_0^j}{(1-z_0^j)^2}$. For $w < 0$, we obtain

$$\left|\frac{G_d(z_0)}{G_d(z)}\right|^2 \geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+A(1-\cos(tj))\right),$$

and for $w \geq 0$, we obtain

$$\begin{aligned}\left|\frac{G_d(z_0)}{G_d(z)}\right|^2 &\geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\frac{1+\frac{2z_0^j(1-\cos(tj))}{(1-z_0^j)^2}}{1+\frac{2wz_0^j(1-\cos(tj))}{(1-z_0^j)^2}}\\ &= \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+\frac{(1-w)\frac{2z_0^j}{(1-z_0^j)^2}(1-\cos(tj))}{1+w\frac{2z_0^j}{(1-z_0^j)^2}(1-\cos(tj))}\right)\\ &\geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+\frac{(1-w)\frac{2z_0^j}{(1-z_0^j)^2}(1-\cos(tj))}{1+2w\frac{2z_0^j}{(1-z_0^j)^2}}\right)\\ &\geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+\frac{(1-w)A}{1+2wA}(1-\cos(tj))\right).\end{aligned}$$

It follows that there exists a constant $B = B(\delta)$ such that

$$\left|\frac{G_d(z_0)}{G_d(z)}\right|^2 \geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\left(1+B(1-\cos(tj))\right)$$

holds uniformly in u. Now it is easy to estimate this product. If $n^{-5/7}\leq |t|\leq \frac{\pi}{2\sqrt{n}}$, we have $n^{-3/14}\leq |tj|\leq \pi$ for all $\sqrt{n}\leq j\leq 2\sqrt{n}$. Then we may make use of the inequality $\frac{u^2}{2}\geq 1-\cos u\geq \frac{2u^2}{\pi^2}$ $(0\leq u\leq \pi)$ to deduce

$$\prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\Big(1+B(1-\cos(tj))\Big) \geq \prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\Big(1+C_1t^2j^2\Big) \geq (1+C_1t^2n)^{\sqrt{n}+O(1)}$$

for some constant $C_1 = C_1(\delta)$. Furthermore, there is a constant $K = K(\delta)$ such that $1+u\geq e^{Ku}$ holds for all $0\leq u\leq C_1\pi^2/4$. Since we have $C_1t^2n\leq C_1\pi^2/4$, it follows that

$$\prod_{\sqrt{n}\leq j\leq 2\sqrt{n}}\Big(1+B(1-\cos(tj))\Big) \geq e^{C_1Kt^2n^{3/2}+O(t^2n)} \geq e^{C_2t^2n^{3/2}}$$

holds for some constant $C_2 = C_2(\delta)$. So if $n^{-5/7} \le |t| \le \frac{\pi}{2\sqrt{n}}$, we obtain the estimate

$$\left|\frac{G_d(z_0)}{G_d(z)}\right| \ge e^{\frac{C_2}{2}t^2n^{3/2}} \ge e^{\frac{C_2}{2}n^{1/14}}, \tag{6.2}$$

which grows faster than any power of n. For $\frac{\pi}{2\sqrt{n}} \le |t| \le \pi$, we consider the set

$$\left|\{\sqrt{n} \le j \le 2\sqrt{n} : |tj - 2k\pi| \le n^{-1/12} \text{ for some integer } k\}\right|.$$

There are at most $\frac{\sqrt{n}|t|}{2\pi} + 1$ possible values for k, since we have $|t|\sqrt{n} \le |tj| \le 2|t|\sqrt{n}$. Furthermore, there are at most $\frac{2n^{-1/12}}{|t|} + 1$ different values of j belonging to any given k. Hence the cardinality of this set is at most

$$\Big(\frac{2n^{-1/12}}{|t|} + 1\Big)\Big(\frac{\sqrt{n}|t|}{2\pi} + 1\Big) = \frac{\sqrt{n}|t|}{2\pi} + O(n^{5/12}) \le \frac{\sqrt{n}}{2} + O(n^{5/12}),$$

so that we obtain the estimate

$$\begin{aligned}\prod_{\sqrt{n}\le j\le 2\sqrt{n}} \Big(1 + B(1-\cos(tj))\Big) &\ge (1 + B(1-\cos n^{-1/12}))^{\sqrt{n}/2+O(n^{5/12})} \\ &= (1 + Bn^{-1/6}/2 + O(n^{-1/3}))^{\sqrt{n}/2+O(n^{5/12})} \\ &= \exp(Bn^{1/3}/4 + O(n^{1/4}))\end{aligned}$$

and finally, for all $\frac{\pi}{2\sqrt{n}} \le |t| \le \pi$,

$$\left|\frac{G_d(z_0)}{G_d(z)}\right| \ge e^{Bn^{1/3}/8+O(n^{1/4})} \ge e^{\frac{C_2}{2}n^{1/14}} \tag{6.3}$$

if C_2 is chosen appropriately. Now we are ready to apply the saddle-point method: we have

$$Q_n(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} z_0^{-n}e^{-int}G_d(z_0e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(g_d(z_0e^{it}))\,dt.$$

This integral is split into two parts, one of which is negligible. In fact we have, by the previous estimates,

$$\begin{aligned}\frac{1}{2\pi}\Big|\int_{n^{-5/7}}^{\pi} \exp(g_d(z_0e^{it}))\,dt\Big| &\le \frac{1}{2\pi}\int_{n^{-5/7}}^{\pi} |\exp(g_d(z_0e^{it}))|\,dt \\ &= \frac{1}{2\pi}\int_{n^{-5/7}}^{\pi} \exp(g_d(z_0))\Big|\frac{G_d(z_0e^{it})}{G_d(z_0)}\Big|\,dt \\ &\le \frac{1}{2}\exp(g_d(z_0))\exp(-C_2n^{1/14}/2),\end{aligned}$$

and an analogous estimate holds for the integral between $-\pi$ and $-n^{-5/7}$ as well. Thus we are left with the integral between $-n^{-5/7}$ and $n^{-5/7}$. The Taylor expansion of $g_d(z)$ around z_0 gives us

$$g_d(z) = g_d(z_0) + g''(z_0)(z-z_0)^2 + O_\delta(n^2(z-z_0)^3).$$

For $z = z_0e^{it}$ with $-n^{-5/7} \le t \le n^{-5/7}$, we have

$$z - z_0 = z_0(it + O(t^2)) = (1 + O_\delta(n^{-1/2}))(it + O(n^{-10/7})) = it + O_\delta(n^{-17/14})$$

and hence

$$g_d(z) = g_d(z_0) - \frac{g''(z_0)}{2}t^2 + O_\delta(n^{-1/7}). \tag{6.4}$$

Now we can consider the remaining part of the integral: we have

$$\frac{1}{2\pi}\int_{-n^{-5/7}}^{n^{-5/7}} \exp g_d(z_0e^{it})\,dt = \frac{1}{2\pi}\exp(g_d(z_0))(1+O_\delta(n^{-1/7}))\int_{-n^{-5/7}}^{n^{-5/7}} e^{-\frac{g''(z_0)}{2}t^2}\,dt,$$

where

$$\begin{aligned}\int_{n^{-5/7}}^{\infty} e^{-\frac{g''(z_0)}{2}t^2}\,dt &\le \int_{n^{-5/7}}^{\infty} e^{-\frac{g''(z_0)}{2}n^{-5/7}t}\,dt \\ &= \int_{n^{-5/7}}^{\infty}\exp\Big(-\Big(\frac{n^{11/14}}{b}+o_\delta(n^{11/14})\Big)t\Big)\,dt \\ &= \big(bn^{-11/14}+o_\delta(n^{-11/14})\big)\exp\left(-\frac{1}{b}n^{1/14}+o_\delta(n^{1/14})\right) \\ &= O_\delta\left(\exp\left(-C_3n^{1/14}\right)\right)\end{aligned}$$

holds uniformly in u for some positive constant $C_3 = C_3(\delta)$. Therefore,

$$\begin{aligned}\int_{-n^{-5/7}}^{n^{-5/7}} e^{-\frac{g''(z_0)}{2}t^2}\,dt &= \int_{-\infty}^{\infty} e^{-\frac{g''(z_0)}{2}t^2}\,dt - 2\int_{n^{-5/7}}^{\infty} e^{-\frac{g''(z_0)}{2}t^2}\,dt \\ &= \sqrt{\frac{2\pi}{g''(z_0)}} + O_\delta\left(\exp\left(-C_3n^{1/14}\right)\right) \\ &= \sqrt{\frac{\pi b}{n^{3/2}}}(1+O_\delta(n^{-1/2})).\end{aligned}$$

So this part of the integral is given by

$$\begin{aligned}\frac{1}{2\pi}\int_{-n^{-5/7}}^{n^{-5/7}}\exp(g_d(z_0e^{it}))\,dt &= \sqrt{\frac{b}{4\pi n^{3/2}}}(1+O_\delta(n^{-1/7}))\exp g_d(z_0) \\ &= \frac{b}{2\pi\sqrt{2u}}\cdot n^{-1}\cdot\exp(2b\sqrt{n})(1+O_\delta(n^{-1/7})).\end{aligned}$$

The remaining part of the integral is, as we have seen, negligible. Hence the asymptotic formula

$$Q_n(u) = \frac{b(u)}{2\pi\sqrt{2u}}\cdot n^{-1}\cdot\exp(2b(u)\sqrt{n})(1+O_\delta(n^{-1/7}))$$

holds uniformly in u. Now, let α_n again be the number of ascents of size d or more in a random partition of n, and set $M_n(t) = \mathbb{E}(e^{(\alpha_n-\mu_n)t/\sigma_n})$, where t is real and μ_n, σ_n are given by $\mu_n = \frac{\sqrt{6n}}{d\pi}$ and $\sigma_n^2 = \frac{(d\pi^2-6)\sqrt{6n}}{2d^2\pi^3}$. Then we have

$$\begin{aligned}M_n(t) &= e^{-\mu_nt/\sigma_n}\frac{Q_n(e^{t/\sigma_n})}{Q_n(1)} \\ &= \frac{\sqrt{6}b(e^{t/\sigma_n})}{\pi}\exp\left(-\frac{\mu_nt}{\sigma_n}-\frac{t}{2\sigma_n}+2\left(b(e^{t/\sigma_n})-\frac{\pi}{\sqrt{6}}\right)\sqrt{n}\right)(1+O_\delta(n^{-1/7})).\end{aligned}$$

It is easy to compute the Taylor expansion of $b(e^t)$ around 0:

$$b(u) = \frac{\sqrt{6}}{\pi} + \frac{\sqrt{6}}{2d\pi}t + \frac{\sqrt{6}(d\pi^2-6)}{8d^2\pi^3}t^2 + O(t^3),$$

from which we obtain

$$M_n(t) = e^{t^2/2}\left(1 + O\left(n^{-1/7} + (|t|+|t|^3)n^{-1/4}\right)\right)$$

uniformly in t as $tn^{-1/12} \to 0$. By Curtiss' theorem [4], the distribution of α_n is asymptotically Gaussian: the normalised random variable $\omega_n := \frac{\alpha_n - \mu_n}{\sigma_n}$ satisfies

$$\mathbb{P}(\omega_n \le x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-t^2/2}\,dt + o(1) \tag{6.5}$$

uniformly for all x as $n \to \infty$. Now, take $T = n^{1/12}/(\log n)$. By Markov's inequality, we have

$$\begin{aligned}\mathbb{P}(\omega_n \ge x) &= \mathbb{P}(e^{\omega_n t} \ge e^{tx}) \le e^{-tx}M_n(t)\\ &= e^{-tx+t^2/2}\left(1 + O\left(n^{-1/7} + (|t|+|t|^3)n^{-1/4}\right)\right)\end{aligned}$$

for arbitrary t. Specialising $t = x$, we obtain

$$\mathbb{P}(\omega_n \ge x) \le e^{-x^2/2}\left(1 + O\left(n^{-1/7} + |T|^3 n^{-1/4}\right)\right)$$

for every $0 \le x \le T$, which simplifies to

$$\mathbb{P}(\omega_n \ge x) \le e^{-x^2/2}\left(1 + O((\log n)^{-3})\right). \tag{6.6}$$

An analogous inequality holds for $\mathbb{P}(\omega_n \le -x)$. If $x \ge T$, we take $t = T$ and obtain

$$\mathbb{P}(\omega_n \ge x) \le e^{-Tx/2}\left(1 + O((\log n)^{-3})\right). \tag{6.7}$$

Summing up, we have the following theorem:

Theorem 5. *α_n, the number of ascents of size d or more in a random partition of n, asymptotically follows a Gaussian distribution with mean $\mathbb{E}(\alpha_n) \sim \mu_n = \frac{\sqrt{6n}}{d\pi}$ and variance $\mathbb{V}(\alpha_n) \sim \sigma_n^2 = \frac{(d\pi^2-6)\sqrt{6n}}{2d^2\pi^3}$. The normalised random variable $\omega_n = \frac{\alpha_n-\mu_n}{\sigma_n}$ satisfies*

$$\mathbb{P}(\omega_n \le x) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-t^2/2}\,dt + o(1)$$

uniformly for all x as $n \to \infty$. Furthermore, the exponential bounds

$$\mathbb{P}(\omega_n \ge x) \le \begin{cases} e^{-x^2/2}\left(1 + O((\log n)^{-3})\right) & 0 \le x \le n^{1/12}/(\log n),\\ e^{-n^{1/12}x/(2\log n)}\left(1 + O((\log n)^{-3})\right) & x \ge n^{1/12}/(\log n)\end{cases}$$

hold as well as the analogous inequalities for $\mathbb{P}(\omega_n \le -x)$.

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