A NOTE ON STIRLING SERIES

MARKUS KUBA AND HELMUT PRODINGER

ABSTRACT. We study sums $S = S(d,n,k) = \sum_{j\ge1} \frac{{j \brack d}}{j^k\binom{n+j}{j}j!}$ with $d \in \mathbb{N} = \{1,2,\dots\}$ and $n,k \in \mathbb{N}_0 = \{0,1,2,\dots\}$ and relate them with (finite) multiple zeta functions. Further, we relate sums S to Nielsen's polylogarithm.

1. Introduction

The unsigned Stirling numbers of the first kind, also called Stirling cycle numbers, are defined by the recurrence relation

$$ {n \brack k} = (n-1){n-1 \brack k} + {n-1 \brack k-1}, \quad n \ge 1, \quad \text{with} \quad {n \brack 0} = \delta_{n,0}, \quad n \ge 0, $$

where $\delta_{i,j}$ denotes the Kronecker delta function. Throughout this work we use Knuth's notation ${n \brack k}$. It is well known that Stirling numbers of the first kind are closely related to harmonic numbers, i.e. ${n \brack 2} = (n-1)!H_{n-1}$, ${n \brack 3} = (n-1)!(H_{n-1}^2 - H_n^{(2)})/2$, where for $s,n \in \mathbb{N}$ the values $H_n^{(s)} = \sum_{\ell=1}^{n} 1/\ell^s$ denote n-th harmonic numbers of order s, $H_n = H_n^{(1)}$. Furthermore, it is known (i.e. see Adamchik [1]) that Stirling numbers of the first kind are expressible in terms of (finite) multiple zeta functions defined by

$$ \zeta_N(a_1,\dots,a_l) = \sum_{N\ge n_1>n_2>\dots>n_l\ge1} \frac{1}{n_1^{a_1}n_2^{a_2}\dots n_\ell^{a_\ell}}, $$

$$ \zeta(a_1,\dots,a_\ell) = \sum_{n_1>n_2>\dots>n_\ell\ge1} \frac{1}{n_1^{a_1}n_2^{a_2}\dots n_\ell^{a_\ell}}, $$

by the following formula

$$ {n \brack k} = (n-1)!\zeta_{n-1}(\underbrace{1,\dots,1}_{k-1}) = (n-1)!\cdot\zeta_{n-1}(\{1\}_{k-1}). $$

Note that for $n,s \in \mathbb{N}_0$ we have $\zeta_n(s) = H_n^{(s)}$. We are interested in evaluations of sums $S = \sum_{j\ge1} \frac{{j \brack d}}{j^k\binom{n+j}{j}j!}$ with $d \in \mathbb{N} = \{1,2,\dots\}$ and $n,k \in \mathbb{N}_0 = \{0,1,2,\dots\}$. We assume that n and k are choosen in way such that $n+k \ge 1$ in order to ensure that the sum converges. Special instances of this family of sums have been studied by Adamchik [1], and also by Choi and Srivastava [5].

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2. Evaluation of sum S

We obtain the following result.

Theorem 1. *The sum $S = S(d, n, k)$ with $d \in \mathbb{N} = \{1, 2, \dots\}$ and $n, k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ can be evaluated in terms of harmonic numbers and (finite) multiple zeta functions,*

$$S = \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \sum_{\sum_{i=1}^{k+1-m} i \cdot m_i = k+1-m} \prod_{r=1}^{k+1-m} \frac{(H_n^{(r)})^{m_r}}{r^{m_r} m_r!}$$
$$+ (-1)^k \sum_{h=1}^{k} \sum_{1 \le \ell_1 < \ell_2 < \cdots < \ell_{h-1} < k} \zeta_n(\ell_1, \ell_2 - \ell_1, \dots, \ell_{h-1} - \ell_{h-2}, d + k - l_{h-1}),$$

subject to $\ell_0 := 0$. We have the short equivalent expression

$$S = (-1)^k \zeta_n^*(\{1\}_{k-1}, d+1) + \sum_{m=2}^{k+1} (-1)^{k+1-m} \zeta(m, \{1\}_{d-1}) \zeta_n^*(\{1\}_{k+1-m}).$$

Remark 1. The second expression for the sum S is given according to a variant of finite multiple zeta functions, $\zeta_N^*(a_1, \dots, a_k)$, which recently attracted some interest,[2, 11, 8, 6] where the summation indices satisfy $N \ge n_1 \ge n_2 \ge \cdots \ge n_k \ge 1$ in contrast to $N \ge n_1 > n_2 > \cdots > n_k > 1$, as in the usual definition (1),

$$\zeta_N^*(a_1, \dots, a_k) = \sum_{N \ge n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{a_1} n_2^{a_2} \dots n_k^{a_k}}.$$

The form stated above is due to the conversion formula below applied to $\zeta_n^*(\{1\}_{k-1}, d+1)$,

$$\zeta_N^*(a_1, \dots, a_k) = \sum_{h=1}^{k} \sum_{\substack{1 \le \ell_1 < \ell_2 < \cdots < \ell_{h-1} < k \\ \ell_0 = 0}} \zeta_N\Bigg(\sum_{i_1=1}^{\ell_1} a_{i_1}, \sum_{i_2=\ell_1+1}^{\ell_2} a_{i_2}, \dots, \sum_{i_h=\ell_{h-1}+1}^{k} a_{i_h} \Bigg).$$

Note that the first term $h = 1$ should be interpreted as $\zeta_N(\sum_{i_1=\ell_0+1}^{k} a_{i_1})$, subject to $\ell_0 = 0$. The notation $\zeta_N^*(a_1, \dots, a_k)$ is chosen in analogy with Aoki and Ohno [2] where infinite counterparts of $\zeta_N^*(a_1, \dots, a_k)$ have been treated; see also Ohno [11].

Remark 2. The sum $\zeta(m, \{1\}_{d-1})$ can be completely transformed into single zeta values. By results of Borwein, Bradley and Broadhoarst [3]

$$\zeta(2, \{1\}_d) = \zeta(d+2)$$
$$\zeta(3, \{1\}_d) = \frac{d+2}{2} \zeta(d+3) - \frac{1}{2} \sum_{\ell=1}^{d} \zeta(\ell+1) \zeta(d+2-\ell).$$

Furthermore, in the general case of $\zeta(m+2, \{1\}_d) = \zeta(d+2, \{1\}_m)$ one obtains products of up to $\min\{m+1, d+1\}$ zeta values, according to the generating function, see [3],

$$\sum_{m,n \ge 0} \zeta(m+2, \{1\}_n) x^{m+1} y^{n+1} = 1 - \exp\Bigg(\sum_{k \ge 2} \frac{x^k + y^k - (x+y)^k}{k} \zeta(k) \Bigg). \qquad (1)$$

Below we state three specific evaluations of the sum S for special choices of d, n, k.

Corollary 1. *For $k = 0$ and arbitrary $n, d \in \mathbb{N}$ we get*

$$S(d,n,0) = \sum_{j\ge 1} \frac{\left[{j \atop d}\right]}{\binom{n+j}{j} j!} = \frac{1}{n^d}.$$

For $k = 1$ and arbitrary $n, d \in \mathbb{N}$ we get

$$S(d,n,1) = \sum_{j\ge 1} \frac{\left[{j \atop d}\right]}{j\binom{n+j}{j} j!} = \zeta(2,\{1\}_{d-1}) - \zeta_n(d+1) = \zeta(d+1) - H_n^{(d+1)}, \tag{2}$$

For $n = 0$ and arbitrary $d, k \in \mathbb{N}$ we get

$$S(d,0,k) = \zeta(k+1,\{1\}_{d-1}).$$

In order to prove the results above we proceed as follows: Since

$$\frac{1}{\binom{n+j}{j}} = \frac{n!}{(n+j)^{\underline{n}}} = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{j+\ell},$$

we obtain

$$S = \sum_{j\ge 1} \frac{\left[{j \atop d}\right]}{j^k\binom{n+j}{j} j!} = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{j\ge 1}\frac{\left[{j \atop d}\right]}{j!j^k(j+\ell)}.$$

We use partial fraction decomposition and obtain

$$\frac{1}{j^k(j+\ell)} = \sum_{m=2}^{k}\frac{(-1)^{k-m}}{j^m\ell^{k+1-m}} + \frac{(-1)^{k+1}}{\ell^k}\Big(\frac{1}{j} - \frac{1}{j+\ell}\Big).$$

Consequently, we get by using the partial fraction decomposition above and the representation of Stirling numbers by finite multiple zeta functions

$$S = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{m=2}^{k+1}\frac{(-1)^{k+1-m}}{\ell^{k+2-m}}\sum_{j\ge 1}\frac{\zeta_{j-1}(\{1\}_{d-1})}{j^m}$$
$$+ \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\frac{(-1)^k}{\ell^{k+1}}\sum_{j\ge 1}\zeta_{j-1}(\{1\}_{d-1})\Big(\frac{1}{j} - \frac{1}{j+\ell}\Big) = S_1 + S_2.$$

By definition of the multiple zeta function we get

$$S_1 = \sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}(-1)^{\ell-1}\sum_{m=2}^{k+1}\frac{(-1)^{k+1-m}}{\ell^{k+2-m}}\zeta(m,\{1\}_{d-1})$$
$$= \sum_{m=2}^{k+1}(-1)^{k+1-m}\zeta(m,\{1\}_{d-1})\sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{\ell^{k+2-m}}.$$

We rewrite the inner sum as

$$\sum_{\ell=1}^{n} n\binom{n-1}{\ell-1}\frac{(-1)^{\ell-1}}{\ell^{k+2-m}} = \sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^{k+1-m}}.$$

This sum can be evaluated by using the following result of Flajolet and Sedgewick [7],

$$\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^m}=\sum_{\sum_{i=1}^m i\cdot m_i=m}\prod_{r=1}^{m}\frac{\big(H_n^{(r)}\big)^{m_r}}{r^{m_r}m_r!};$$

we recall that $H_n^{(s)}=\sum_{\ell=1}^n 1/\ell^s$ denotes the n-th harmonic number of order s; in other words we have $H_n^{(s)}=\zeta_n(s)$, according to our previous definition of finite multiple zeta functions (1). The multiple zeta function $\zeta(m,\{1\}_d)$ is evaluated using a result of Borwein, Bradley and Broadhoarst [3], see Remark 2. Consequently, we can write sum S_1 as a finite sum involving higher order harmonic numbers and products of zeta functions and obtain the first part of our result. For the simplification of the inner sum

$$S_2=\sum_{\ell=1}^{n}n\binom{n-1}{\ell-1}(-1)^{\ell-1}\frac{(-1)^k}{\ell^{k+1}}\sum_{j\geq 1}\zeta_{j-1}(\{1\}_{d-1})\big(\frac{1}{j}-\frac{1}{j+\ell}\big),$$

we use the notation $T_{m,\ell}=\sum_{j\geq1}\zeta_{j-1}(\{1\}_m)\big(\frac{1}{j}-\frac{1}{j+\ell}\big)$. Subsequently, we interchange summation, compare with Panholzer and Prodinger [10]. First we start with the simple case $m=1$ and calculate $T_{1,\ell}$, since it is most instructive.

$$T_{1,\ell}=\sum_{j\geq1}H_{j-1}\Big(\frac{1}{j}-\frac{1}{j+\ell}\Big)=\sum_{j\geq1}H_j\Big(\frac{1}{j+1}-\frac{1}{j+1+\ell}\Big).$$

Since by definition $H_j=\sum_{h=1}^j 1/h$ we obtain after summation change (partial summation)

$$T_{1,\ell}=\sum_{h\geq1}\frac{1}{h}\sum_{j\geq h}\Big(\frac{1}{j+1}-\frac{1}{j+1+\ell}\Big)=\sum_{h\geq1}\frac{1}{h}\sum_{j=1}^{\ell}\frac{1}{j+h}.$$

By partial fraction decomposition we get

$$T_{1,\ell}=\sum_{j=1}^{\ell}\frac{1}{j}\sum_{h\geq1}\Big(\frac{1}{h}-\frac{1}{j+h}\Big)=\sum_{j=1}^{\ell}\frac{H_j}{j}=\frac{H_\ell^2+H_\ell^{(2)}}{2}.$$

Now we turn to the general case $T_{m,\ell}$. Shifting the index as before, and changing the order of summation leads to

$$T_{m,\ell}=\sum_{h\geq1}\frac{\zeta_{h-1}(\{1\}_{m-1})}{h}\sum_{j\geq h}\Big(\frac{1}{j+1}-\frac{1}{j+1+\ell}\Big)$$

Consequently,

$$T_{m,\ell}=\sum_{j=1}^{\ell}\frac{1}{j}\sum_{h\geq1}\zeta_{h-1}(\{1\}_{m-1})\Big(\frac{1}{h}-\frac{1}{h+j}\Big)=\sum_{j=1}^{\ell}\frac{1}{j}T_{m-1,j}.$$

Hence, the value $T_{m,\ell}$ is a variant of the finite multiple zeta function $\zeta_\ell(\{1\}_{m+1})$, where the summation indices satisfy $N\geq n_1\geq n_2\geq\cdots\geq n_m\geq n_{m+1}\geq1$ instead of $N\geq n_1>n_2>$

$\dots > n_m > n_{m+1} > 1$, see Remark 1, such that $T_{m,\ell} = \zeta^*_\ell(\{1\}_{m+1})$. We further obtain

$$T_{m,\ell} = \zeta^*_\ell(\{1\}_{m+1}) = \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^{m+1}},$$

according to the well known formula $\binom{n}{k} = \sum_{\ell=k}^{n} \binom{\ell-1}{k-1}$. Consequently, the sum S_2 simplifies to

$$S_2 = (-1)^k \sum_{\ell=1}^{n} \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \sum_{h=1}^{\ell} \binom{\ell}{h} \frac{(-1)^{h-1}}{h^d} = (-1)^k \sum_{h=1}^{n} \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^{n} \binom{n}{\ell}\binom{\ell}{h} \frac{(-1)^{\ell-1}}{\ell^k},$$

or equivalently

$$S_2 = (-1)^k \sum_{\ell=1}^{n} \binom{n}{\ell} \frac{(-1)^{\ell-1}}{\ell^k} \zeta^*_\ell(\{1\}_d).$$

In order to obtain the final form of S_2 for $k \in \mathbb{N}$ we combine our previous considerations as follows:

$$S_2 = (-1)^k \sum_{h_1=1}^{n} \frac{1}{h_1} \sum_{h_2=1}^{h_1} \frac{1}{h_2} \cdots \sum_{h_{k+1}=1}^{h_k} \binom{h_k}{h_{k+1}} (-1)^{h_{k+1}-1} \zeta^*_{h_{k+1}}(\{1\}_d).$$

We use the fact that $\sum_{\ell=h}^{n} \binom{n}{\ell}\binom{\ell}{h}(-1)^{\ell-1} = \delta_{h,n}(-1)^{n-1}$ and the sum S_2 simplifies to

$$S_2 = (-1)^k \zeta^*_n(\{1\}_{k-1}, d+1).$$

In the case $k = 0$ we use

$$S_2 = \sum_{h=1}^{n} \frac{(-1)^{h-1}}{h^d} \sum_{\ell=h}^{n} \binom{n}{\ell}\binom{\ell}{h} (-1)^{\ell-1} = \frac{1}{n^d}.$$

3. Relation to Nielsen's polylogarithm

Nielsen's polylogarithm $L_{k,d}(z)$ is defined by

$$L_{k,d}(z) = \frac{(-1)^{k-1+d}}{(k-1)!d!} \int_0^1 \frac{\log^{k-1}(t)\log^d(1-zt)}{t} dt$$

By definition of the generating function of the Stirling cycle numbers

$$\sum_{n\ge k} {n \brack k} \frac{z^n}{n!} = \frac{(-1)^k \log^k(1-z)}{k!},$$

it is evident that $L_{k,d}(z) = \sum_{j\ge1} \frac{{j \brack d} z^j}{j^k j!}$. Hence, we obtain the following result.

Proposition 1. *The series $S(z) = S_{d,n,k}(z) = \sum_{j\ge1} \frac{{j \brack d} z^j}{j^k \binom{n+j}{j} j!}$ can be expressed by Nielsen's polylogarithm $L_{k,d}(z)$ in the following way.*

$$\sum_{j\ge1} \frac{{j \brack d} z^j}{j^k \binom{n+j}{j} j!} = \frac{n}{z} \int_0^z \left(1 - \frac{u}{z}\right)^{n-1} L_{k,d}(u) du.$$

Note that

$$S_{d,n,k}(z) = \sum_{\ell=1}^{n} \ell(-1)^{\ell-1}\binom{n}{\ell}\frac{(-1)^{k-1}}{(k-1)!d!}\frac{1}{z^l}\int_0^z u^{\ell-1}\int_0^1 \frac{\log^{k-1}(t)\log^d(1-ut)}{t}dtdu$$
$$= \sum_{\ell=1}^{n} \ell(-1)^{\ell-1}\binom{n}{\ell}\frac{1}{z^l}\int_0^z u^{\ell-1}L_{k,d}(u)du.$$

Interchanging summation and integration gives the desired result.

3.1. **Generalized r-Stirling numbers of the first kind.** In a recent work Mező [9] considered series involving so-called r-Stirling numbers of the first kind, see Broder [4]. For any positive integer $r \in \mathbb{N}$ the quantity $\left[{n \atop m}\right]_r$ denotes the number of permutations of the set $\{1,\dots,n\}$ having m cycles such that the first r element are in distinct cycles. These numbers obey the recurrence relation

$$\left[{n \atop k}\right]_r = (n-1)\left[{n-1 \atop k}\right]_r + \left[{n-1 \atop k-1}\right]_r, \quad n > r, \quad \left[{n \atop k}\right]_r = \delta_{k,r}, \quad n = r, \quad \left[{n \atop k}\right]_r = 0, \quad n < r.$$

For $r = 0$ and $r = 1$ these numbers coincide with the ordinary Stirling numbers of the first kind. We will consider the series

$$S^{(r)}(z) = S^{(r)}_{d,n,k,\ell}(z) = \sum_{j\geq 1}\frac{\left[{j+\ell+r \atop d+r}\right]_r z^j}{j^k\binom{n+j}{j}j!},$$

which generalizes the series considered by Mező [9] (case $n = 0$) and our previously considered series S (case $\ell = r = 0$). Subsequently, we obtain representations of $S^{(r)}_{d,n,k,0}(z)$ and also of $S^{(r)}_{d,n,k,\ell}(z)$. We introduce the quantity $L^{(r)}_{n,k}(z)$, which generalizes Nielsen's polylogarithm

$$L^{(r)}_{k,d}(z) = \frac{(-1)^{k-1+d}}{(k-1)!d!}\int_0^1 \frac{\log^{k-1}(t)\log^d(1-zt)}{(1-zt)^r t}dt.$$

Proposition 2. *The series* $S^{(r)}_{d,n,k,0}(z) = \sum_{j\geq 1}\frac{\left[{j+r \atop d+r}\right]_r z^j}{j^k\binom{n+j}{j}j!}$ *can be expressed by* $L^{(r)}_{k,d}(z)$ *in the following way.*

$$S^{(r)}_{d,n,k,0}(z) = \frac{n}{z}\int_0^z \Big(1-\frac{u}{z}\Big)^{n-1} L^{(r)}_{k,d}(u)du.$$

The series $S^{(r)}_{d,n,k,\ell}(z)$ *can be expression as a linear combination of the sums* $S^{(r+\ell)}_{h,n,k,0}(z)$, *with* $0 \leq h \leq d$.

First we note that the r-Stirling numbers of the first kind have the generating function

$$\sum_{n\geq k}\left[{n+r \atop k+r}\right]_r \frac{z^n}{n!} = \frac{(-1)^k\log^k(1-z)}{k!(1-z)^r}.$$

We observe that

$$L^{(r)}_{k,d}(z) = \sum_{j\geq 1}\frac{\left[{j+r \atop d+r}\right]_r z^j}{j^k\binom{n+j}{j}j!} = S^{(r)}_{d,0,k,0}(z).$$

Consequently, we get

$$S^{(r)}_{d,n,k,0}(z) = \sum_{j\ge1} \frac{\genfrac{[}{]}{0pt}{}{j+r}{d+r}_r z^j}{j^k\binom{n+j}{j} j!} = \int_0^z \frac{n(1-\frac{u}{z})^n}{(z-u)} L^{(r)}_{k,d}(u)du.$$

Next we turn to the general case $\ell \in \mathbb{N}$. Since

$$\sum_{n\ge k} \begin{bmatrix} n+r \\ d+r \end{bmatrix}_r \frac{z^n}{n!} = \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r},$$

we obtain the exponential generating function of $\genfrac{[}{]}{0pt}{}{n+\ell+r}{d+r}_r$ by differentiating $\frac{(-1)^d \log^d(1-z)}{d!(1-z)^r}$ ℓ-times with respect to z and a subsequent shift of the index,

$$\frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} = \sum_{n\ge d+\ell} \begin{bmatrix} n+r \\ d+r \end{bmatrix}_r \frac{z^{n-\ell}}{(n-\ell)!} = \sum_{n\ge \max\{d-\ell,0\}} \begin{bmatrix} n+\ell+r \\ d+r \end{bmatrix}_r \frac{z^n}{n!}.$$

By Faà di Bruno's formula we get

$$\frac{\partial^\ell}{\partial z^\ell} \frac{(-1)^d \log^d(1-z)}{d!(1-z)^r} = \sum_{h=0}^{\ell} \frac{d^{\underline{h}}(-1)^h \log^{d-h}(1-z)}{(1-z)^{r+\ell}} \sum_{i=h}^{\ell} r^{\overline{\ell-i}} B_{i,h}(0!, 1!, 2!, \dots, (i-h)!),$$

where $B_{i,h}(x_1, x_2, \dots, x_{i-h+1})$ denote the Bell polynomials. Consequently, we can express the sum $S^{(r)}_{d,n,k,\ell}(z)$ as a linear combination of the sums $S^{(r)}_{h,n,k,0}(z)$, with $0 \le h \le d$, which proves the stated result.

Remark 3. Note that the sums $S^{(r)}_{d,n,k,\ell}(1) = \sum_{j\ge1} \frac{\genfrac{[}{]}{0pt}{}{j+\ell+r}{d+r}_r z^j}{j^k\binom{n+j}{j} j!}$ can in principle also be treated using our previous approach; however, the expression become much more involved, therefore we refrain from going into this matter. Furthermore, one can evaluate sums of the form $\sum_{j\ge1} \frac{\genfrac{[}{]}{0pt}{}{j}{d}}{j^k\binom{n+j}{j}^g j!}$, with $g \in \mathbb{N}$; however, the expressions get more and more involved.

Historical remark

The author H.P. has found the formula (2) empirically in 2003. He contacted several specialists about it and got feedback from Christian Krattenthaler who provided a *hypergeometric* proof for it. Eventually it turned out that it was known already [5]. We are happy that in 2009 we could put new life into this project.

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Markus Kuba, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8-10/104, 1040 Wien, Austria

E-mail address: kuba@dmg.tuwien.ac.at

Helmut Prodinger, Department of Mathematics, University of Stellenbosch, 7602 Stellenbosch, South Africa

E-mail address: hproding@sun.ac.za