q–ENUMERATION OF SALIÉ PERMUTATIONS

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ABSTRACT. Salié permutations are defined to be first alternating, and then monotone. We q-enumerate them by considering words (instead of permutations), equipped with geometric probabilities.

1. Introduction

Salié permutations $\pi = x_1 \dots x_{2n}$ are characterized by the property that, for some $1 \le k \le n$, $x_1 \dots x_{2k}$ is *alternating* ($x_1 < x_2 > x_3 < \cdots$) and $x_{2k} \dots x_{2n}$ is *monotone* ($x_{2k} < \cdots < x_{2n}$).

They are enumerated for $n \ge 1$ by

$$(2n)![z^{2n}]\frac{1}{2}\frac{\cosh z}{\cos z},$$

or, more precisely, for $n \ge 0$ by

$$(2n)![z^{2n}]\Big(\frac{1}{2}\frac{\cosh z}{\cos z} - \frac{1}{2}\Big);$$

see [1] and [2].

In a recent paper [2], q-Salié numbers were introduced by means of the generating function

$$\sum_{n\ge0}\frac{q^n z^{2n}}{(q)_{2n}}\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{(q)_{2n}};$$

3 variants

$$\sum_{n\ge0}\frac{z^{2n}}{(q)_{2n}}\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{(q)_{2n}},$$

$$\sum_{n\ge0}\frac{q^{2n} z^{2n}}{(q)_{2n}}\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{(q)_{2n}},$$

$$\sum_{n\ge0}\frac{q^{n^2} z^{2n}}{(q)_{2n}}\Big/\sum_{n\ge0}\frac{(-1)^n z^{2n}}{(q)_{2n}},$$

were also discussed. (We use the notation $(x)_n = (1-x)\cdots(1-xq^{n-1})$.)

In the present note, we want to introduce *several* versions of q-Salié numbers, by appropriate q-enumeration of permutations (resp. words) following the pattern $<>\cdots<><<\cdots<$. Our model (that has proved to be successful e.g., in an earlier paper [3]) is the one of words over

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the infinite alphabet $\{1, 2, 3, \dots\}$, equipped with geometric probabilities pq^{k-1}, where we write $p = 1 - q$ for convenience. In the limit $q \to 1$, each ordering of the letters $x_1 \dots x_n$ becomes equally likely, and equal letters tend to disappear, whence this approaches the model of *random permutations.* However, for words, we could equally well consider the pattern $\le > \cdots \le > \le \le \cdots \le$ and some others. Furthermore, for the enumerations of permutations it does not make a difference if one would exchange $< \longleftrightarrow >$; for words it *does!* In this way, we naturally come up with 8 versions of q-Salié numbers!

What we are going to compute in the sequel is the probability that a random word of length $2n$ satisfies a prescribed Salié-pattern, like, say, $<> \cdots <> << \cdots <$. Typically, we will work with a generating function $f(u) = f(u, z)$, where the coefficent of $z^n u^i$ is the probability of alternating word (of odd length) with last letter i, and then allow for a last run of increasing letters (an odd number of *up*-steps)). Maintaining the last letter allows to compute an extra *up-down*-double-step: Say, for $<\ge$, this is done via

$$\sum_{j>i} pq^{j-1} \sum_{k\le j} pq^{k-1}u^k = \frac{pu}{(qu)_1}q^i - \frac{p^2qu^2}{(qu)_2}(uq^2)^i.$$

So, there are 2 substitutions, $u \to q$ and $u \to uq^2$ involved. The other cases are similar.

For the *up*-steps at the end, compute, say,

$$\sum_{j>i} pq^{j-1}u^j = \frac{pu}{(qu)_1}(qu)^i.$$

So, for an odd number of steps $<$, we are led to the following

$$S := \sum_{l\ge 0} \frac{p^{2l+1}q^{l(2l+1)}z^{2l+1}}{(q)_{2l+1}} f(q^{2l+1}),$$

and in the instance $\le$

$$S := \sum_{l\ge 0} \frac{p^{2l+1}z^{2l+1}}{q^{2l+1}(q)_{2l+1}} f(q^{2l+1}).$$

In the instance of *up* and *down* exchanged, a run of an odd number of steps $\ge$, say, at the end, leads to the complicated formula

$$\sum_{l \text{ odd}} \frac{z^l}{(q)_l} \sum_{0\le i\le l} p^l q^{\binom{l-i}{2}} (-1)^i \begin{bmatrix} l \\ i \end{bmatrix} f(q^i).$$

This is not really practical. Thus, in this case, we consider the word from *right to left,* i.e., start with a run of up-steps, followed by an alternating word.

2. The instance $\le>$

First, we enumerate words with pattern $\le > \cdots \le >$, which, according to [3], lead to q-tangent functions. Set

$$f(u) = f(u, z) = \sum_{n\ge 1} z^n \sum_{i\ge 1} u^i[\text{probability that a word of odd length } n \text{ has shape } \le > \cdots \le > \text{ and last letter } i].$$

Denote the coefficient of z^{2n+1} in $f(u)$ by $T^{\le>}_{2n+1}(u)$. Then we have the recursion

$$\begin{aligned}T^{\le>}_{2n+1}(u)&=\frac{pu}{q(1-qu)}T^{\le>}_{2n-1}(q)-\frac{p^2}{q^2(1-qu)(1-q^2u)}T^{\le>}_{2n-1}(q^2u),\\ T^{\le>}_{1}(u)&=\frac{pu}{1-qu},\end{aligned}$$

which translates into

$$f(u)=\frac{puz}{1-qu}+\frac{puz^2}{q(1-qu)}f(q)-\frac{p^2z^2}{q^2(1-qu)(1-q^2u)}f(q^2u).$$

Iterating this we find

$$\begin{aligned}f(u)&=\frac{puz}{1-qu}+\frac{puz^2}{q(1-qu)}f(q)\\ &\quad-\frac{p^2z^2}{q^2(1-qu)(1-q^2u)}\left[\frac{pq^2uz}{1-q^3u}+\frac{pq^2uz^2}{q(1-q^3u)}f(q)\right.\\ &\quad-\frac{p^2z^2}{q^2(1-q^3u)(1-q^4u)}\left[\frac{pq^4uz}{1-q^5u}+\frac{pq^4uz^2}{q(1-q^5u)}f(q)\dots\right.\\ &=u\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(qu)_{2k+1}}+f(q)\frac{z}{q}u\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(qu)_{2k+1}}.\end{aligned}\tag{2.1}$$

Plugging in $u=q$ and solving the resulting equation we get

$$f(q)=\frac{q\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(q^2)_{2k+1}}}{1-z\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(q^2)_{2k+1}}}.$$

Using this in (2.1) we get after some simplifications

$$f(u)=u\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(qu)_{2k+1}}\Big/\sum_{k\ge0}\frac{(-1)^kp^{2k}z^{2k}}{(q)_{2k}}.$$

Now, as discussed, in order to model the final odd number of up ($\le$) steps, we must compute

$$\begin{aligned}S^{\le>}(z)&=\sum_{l\ge0}\frac{p^{2l+1}z^{2l+1}}{q^{2l+1}(q)_{2l+1}}f(q^{2l+1})\\ &=\sum_{l\ge0}\frac{p^{2l+1}z^{2l+1}}{(q)_{2l+1}}\sum_{k\ge0}\frac{(-1)^kp^{2k+1}z^{2k+1}}{(q^{2l+1})_{2k+1}}\Big/\sum_{k\ge0}\frac{(-1)^kp^{2k}z^{2k}}{(q)_{2k}}.\end{aligned}$$

It is a bit nicer to consider $S^{\le>}(z/p)$; let us abbreviate

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^kz^{2k}}{(q)_{2k}}.$$

Then

$$S^{\le>}(\tfrac{z}{p})=\frac{1}{\mathcal{N}}\sum_{l\ge0}\sum_{k\ge0}\frac{(-1)^kz^{2k+2l+2}}{(q)_{2k+2l+2}}$$

$$\begin{aligned}
&= \frac{1}{\mathcal{N}} \sum_{l\ge 0} (-1)^{l+1} \sum_{k>l} \frac{(-1)^k z^{2k}}{(q)_{2k}}\\
&= \sum_{l\ge 0} (-1)^{l+1} \biggl[1 - \frac{1}{\mathcal{N}} \sum_{0\le k\le l} \frac{(-1)^k z^{2k}}{(q)_{2k}}\biggr]\\
&= \lim_{\varepsilon\to 1} \sum_{l\ge 0} (-1)^{l+1}\varepsilon^l \biggl[1 - \frac{1}{\mathcal{N}} \sum_{0\le k\le l} \frac{(-1)^k z^{2k}}{(q)_{2k}}\biggr]\\
&= \lim_{\varepsilon\to 1} \biggl[-\frac{1}{1+\varepsilon} + \frac{1}{\mathcal{N}} \sum_{l\ge 0} (-\varepsilon)^l \sum_{0\le k\le l} \frac{(-1)^k z^{2k}}{(q)_{2k}}\biggr]\\
&= -\frac{1}{2} + \frac{1}{\mathcal{N}} \lim_{\varepsilon\to 1} \sum_{k\ge 0} \frac{(-1)^k z^{2k}}{(q)_{2k}} \sum_{l\ge k} (-\varepsilon)^l\\
&= -\frac{1}{2} + \frac{1}{\mathcal{N}} \lim_{\varepsilon\to 1} \sum_{k\ge 0} \frac{(-1)^k z^{2k}}{(q)_{2k}} \frac{(-\varepsilon)^k}{1+\varepsilon}\\
&= -\frac{1}{2} + \frac{1}{2\mathcal{N}} \sum_{k\ge 0} \frac{z^{2k}}{(q)_{2k}}.
\end{aligned}$$

Thus, letting $q \to 1$ in $S^{\le>}(z)$, we get

$$-\frac{1}{2} + \frac{1}{2} \sum_{k\ge 0} \frac{z^{2k}}{(2k)!} \Big/ \sum_{k\ge 0} \frac{(-1)^k z^{2k}}{(2k)!} = -\frac{1}{2} + \frac{1}{2}\frac{\cosh z}{\cos z},$$

as predicted.

The first few values $\bigl((q)_{2n}[z^{2n}]S^{\le>}(z/p)\bigr)$ are

$$1,\ (1+q^2)(1+q+q^2),\ (1+q^2)(1+q+2q^2+4q^3+6q^4+6q^5+6q^6+5q^7+4q^8+2q^9+q^{10}).$$

3. The instance $\le\ge$

We start again with $\le\ge \cdots \le\ge$ and have the recursion

$$\begin{aligned}
T_{2n+1}^{\le\ge}(u) &= \frac{pu}{q(1-qu)} T_{2n-1}^{\le\ge}(q) - \frac{p^2u}{q(1-qu)(1-q^2u)} T_{2n-1}^{\le\ge}(q^2u),\\
T_1^{\le\ge}(u) &= \frac{pu}{1-qu},
\end{aligned}$$

whence

$$f(u) = \frac{puz}{1-qu} + \frac{puz^2}{q(1-qu)} f(q) - \frac{p^2uz^2}{q(1-qu)(1-q^2u)} f(q^2u)$$

and

$$f(u) = \sum_{k\ge 0} \frac{p^{2k+1}z^{2k+1}(-1)^k u^{k+1} q^{k^2}}{(qu)_{2k+1}} + f(q) \sum_{k\ge 0} \frac{p^{2k+1}z^{2k+2}(-1)^k u^{k+1} q^{k^2-1}}{(qu)_{2k+1}}.$$

Plugging in and simplifying:

$$f(q)=\sum_{k\ge0}\frac{p^{2k+1}z^{2k+1}(-1)^kq^{k^2+k+1}}{(q^2)_{2k+1}}\Big/\sum_{k\ge0}\frac{p^{2k}z^{2k}(-1)^kq^{k(k-1)}}{(q)_{2k}}.$$

Using this, we eventually find

$$f(u)=\sum_{k\ge0}\frac{p^{2k+1}z^{2k+1}(-1)^ku^{k+1}q^{k^2}}{(qu)_{2k+1}}\Big/\sum_{k\ge0}\frac{p^{2k}z^{2k}(-1)^kq^{k(k-1)}}{(q)_{2k}}.$$

Again, we must compute

$$\begin{aligned}S^{\le\ge}(z)&=\sum_{l\ge0}\frac{p^{2l+1}z^{2l+1}}{q^{2l+1}(q)_{2l+1}}f(q^{2l+1})\\&=\sum_{l\ge0}\frac{p^{2l+1}z^{2l+1}}{(q)_{2l+1}}\sum_{k\ge0}\frac{p^{2k+1}z^{2k+1}(-1)^kq^{(2l+1)k}q^{k^2}}{(q^{2l+2})_{2k+1}}\Big/\sum_{k\ge0}\frac{p^{2k}z^{2k}(-1)^kq^{k(k-1)}}{(q)_{2k}}.\end{aligned}$$

We consider $S^{\le\ge}(z/p)$ and abbreviate

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^kq^{k(k-1)}z^{2k}}{(q)_{2k}}.$$

Then

$$\begin{aligned}S^{\le\ge}(\tfrac{z}{p})&=\frac{1}{\mathcal{N}}\sum_{l\ge0}\sum_{k\ge0}\frac{(-1)^kz^{2k+2l+2}q^{(2l+1)k+k^2}}{(q)_{2k+2l+2}}\\&=\frac{1}{\mathcal{N}}\sum_{l\ge0}(-1)^{l+1}\sum_{k>l}\frac{(-1)^kz^{2k}q^{k(k-1)-(l+1)l}}{(q)_{2k}}\\&=\frac{1}{\mathcal{N}}\sum_{l\ge0}(-1)^{l+1}q^{-(l+1)l}\Bigg[\mathcal{N}-\sum_{0\le k\le l}\frac{(-1)^kz^{2k}q^{k(k-1)}}{(q)_{2k}}\Bigg].\end{aligned}$$

To let $q\to1$ in $S^{\le\ge}(z)$, we consider

$$\lim_{\varepsilon\to1}\frac{1}{\mathcal{N}(pz)}\sum_{l\ge0}(-1)^{l+1}q^{-(l+1)l}\varepsilon^l\Bigg[\mathcal{N}(pz)-\sum_{0\le k\le l}\frac{(-1)^k(pz)^{2k}q^{k(k-1)}}{(q)_{2k}}\Bigg].$$

Now we can let $q\to1$:

$$\begin{aligned}&\lim_{\varepsilon\to1}\frac{1}{\cos z}\sum_{l\ge0}(-1)^{l+1}\varepsilon^l\Bigg[\cos z-\sum_{0\le k\le l}\frac{(-1)^kz^{2k}}{(2k)!}\Bigg]\\&=-\frac12+\lim_{\varepsilon\to1}\frac{1}{\cos z}\sum_{l\ge0}(-1)^l\varepsilon^l\sum_{0\le k\le l}\frac{(-1)^kz^{2k}}{(2k)!}\\&=-\frac12+\frac{1}{\cos z}\lim_{\varepsilon\to1}\sum_{k\ge0}\frac{(-1)^kz^{2k}}{(2k)!}(-\varepsilon)^k\frac{1}{1+\varepsilon}\end{aligned}$$

$$= -\frac{1}{2} + \frac{1}{2}\frac{\cosh z}{\cos z}.$$

as before (and predicted).

Note also the form

$$S^{\le\ge}(\tfrac{z}{p}) = \frac{1}{\mathcal{N}}\sum_{l\ge1}\frac{(-1)^l z^{2l} q^{l(l-1)}}{(q)_{2l}}\sum_{0\le k<l}(-1)^{l+1}q^{-l(l+1)}.$$

The first few values are

$$1,\ 2+q+q^2+q^3+q^4,\ 3+3q+5q^2+7q^3+9q^4+10q^5+11q^6+9q^7+8q^8+5q^9+3q^{10}+2q^{11}+q^{12}.$$

4. The instance $<\ge$

We start again with $<\ge\cdots<\ge$ and have the recursion

$$T^{<\ge}_{2n+1}(u) = \frac{pu}{q(1-qu)}T^{<\ge}_{2n-1}(q) - \frac{p^2}{q^2(1-qu)(1-q^2u)}T^{<\ge}_{2n-1}(q^2u),$$
$$T^{<\ge}_{1}(u) = \frac{pu}{1-qu},$$

whence

$$f(u) = \frac{puz}{1-qu} + \frac{puz^2}{(1-qu)}f(q) - \frac{p^2qu^2z^2}{(1-qu)(1-q^2u)}f(q^2u)$$

and

$$f(u) = \sum_{k\ge0}\frac{(-1)^k p^{2k+1}z^{2k+1}u^{2k+1}q^{2k^2+k}}{(qu)_{2k+1}} + zf(q)\sum_{k\ge0}\frac{(-1)^k p^{2k+1}z^{2k+1}u^{2k+1}q^{2k^2+k}}{(qu)_{2k+1}},$$

from which we get

$$f(q) = \sum_{k\ge0}\frac{(-1)^k p^{2k+1}z^{2k+1}q^{2k^2+3k+1}}{(q^2)_{2k+1}}\Bigg/\sum_{k\ge0}\frac{(-1)^k p^{2k}z^{2k}q^{k(2k-1)}}{(q)_{2k}}$$

and thus

$$f(u) = \sum_{k\ge0}\frac{(-1)^k p^{2k+1}z^{2k+1}u^{2k+1}q^{2k^2+k}}{(qu)_{2k+1}}\Bigg/\sum_{k\ge0}\frac{(-1)^k p^{2k}z^{2k}q^{2k^2-k}}{(q)_{2k}}.$$

This time we must compute

$$S^{<\ge}(z) = \sum_{l\ge0}\frac{p^{2l+1}q^{l(2l+1)}z^{2l+1}}{(q)_{2l+1}}f(q^{2l+1})$$
$$= \sum_{l\ge0}\frac{p^{2l+1}q^{l(2l+1)}z^{2l+1}}{(q)_{2l+1}}\sum_{k\ge0}\frac{(-1)^k p^{2k+1}z^{2k+1}q^{(2l+1)(2k+1)}q^{2k^2+k}}{(q^{2l+2})_{2k+1}}\Bigg/\sum_{k\ge0}\frac{(-1)^k p^{2k}z^{2k}q^{2k^2-k}}{(q)_{2k}}.$$

We consider $S^{<\ge}(z/p)$ and abbreviate

$$\mathcal{N} := \sum_{k\ge0}\frac{(-1)^k q^{2k^2-k}z^{2k}}{(q)_{2k}}.$$

Then

$$\begin{aligned} S^{<\geq}(\tfrac{z}{p}) &= \frac{1}{\mathcal{N}} \sum_{l\geq 0} q^{l(2l+1)} \sum_{k\geq 0} \frac{(-1)^k z^{2k+2l+2} q^{(2l+1)(2k+1)} q^{2k^2+k}}{(q)_{2k+2l+2}} \\ &= \frac{1}{\mathcal{N}} \sum_{l\geq 0} (-1)^{l+1} \sum_{k>l} \frac{(-1)^k z^{2k} q^{2k^2-k}}{(q)_{2k}} \\ &= \frac{1}{\mathcal{N}} \sum_{l\geq 0} (-1)^{l+1} \Big[\mathcal{N} - \sum_{0\leq k\leq l} \frac{(-1)^k z^{2k} q^{2k^2-k}}{(q)_{2k}} \Big]. \end{aligned}$$

The same procedure as in the section of the instance $\leq>$ leads then to

$$S^{<\geq}(\tfrac{z}{p}) = -\frac{1}{2} + \frac{1}{2\mathcal{N}} \sum_{k\geq 0} \frac{z^{2k} q^{2k^2-k}}{(q)_{2k}}.$$

Letting $q \to 1$ in $S^{<\geq}(z)$ produces again

$$-\frac{1}{2} + \frac{1}{2}\frac{\cosh z}{\cos z}.$$

The first few values are

$$q,\ q^2(1+q^2)(1+q+q^2),\ q^3(1+q^2)(1+2q+4q^2+5q^3+6q^4+6q^5+6q^6+4q^7+2q^8+q^9+q^{10}).$$

5. The instance <>

Here we have

$$\begin{aligned} T^{<>}_{2n+1}(u) &= \frac{pu}{1-qu} T^{<>}_{2n-1}(q) - \frac{p^2u}{(1-qu)(1-q^2u)} T^{<>}_{2n-1}(q^2u), \\ T^{<>}_{1}(u) &= \frac{pu}{1-qu}, \end{aligned}$$

thus

$$f(u) = \frac{puz}{1-qu} + \frac{puz^2}{1-qu} f(q) - \frac{p^2uz^2}{(1-qu)(1-q^2u)} f(q^2u)$$

and

$$f(u) = \sum_{k\geq 1} \frac{(-1)^{k-1} p^{2k-1} q^{k(k-1)} u^k z^{2k-1}}{(qu)_{2k-1}} + f(q) \sum_{k\geq 1} \frac{(-1)^{k-1} p^{2k-1} q^{k(k-1)} u^k z^{2k}}{(qu)_{2k-1}}.$$

Furthermore,

$$f(q) = \sum_{k\geq 1} \frac{(-1)^{k-1} p^{2k} q^{k^2} z^{2k-1}}{(q)_{2k}} \Big/ \sum_{k\geq 0} \frac{(-1)^k p^{2k} q^{k^2} z^{2k}}{(q)_{2k}}$$

and

$$f(u) = \sum_{k\geq 0} \frac{(-1)^k p^{2k+1} q^{k(k+1)} u^{k+1} z^{2k+1}}{(qu)_{2k+1}} \Big/ \sum_{k\geq 0} \frac{(-1)^k p^{2k} q^{k^2} z^{2k}}{(q)_{2k}}.$$

Therefore

$$\begin{aligned}S^{<>}(z) &= \sum_{l\ge0}\frac{p^{2l+1}q^{l(2l+1)}z^{2l+1}}{(q)_{2l+1}}f(q^{2l+1})\\ &= \sum_{l\ge0}\frac{p^{2l+1}q^{l(2l+1)}z^{2l+1}}{(q)_{2l+1}}\sum_{k\ge0}\frac{(-1)^kp^{2k+1}q^{k(k+1)}q^{(2l+1)(k+1)}z^{2k+1}}{(q^{2l+2})_{2k+1}}\Big/\sum_{k\ge0}\frac{(-1)^kp^{2k}q^{k^2}z^{2k}}{(q)_{2k}}.\end{aligned}$$

Set

$$\mathcal{N} := \sum_{k\ge0}\frac{(-1)^kq^{k^2}z^{2k}}{(q)_{2k}},$$

then

$$\begin{aligned}S^{<>}(\tfrac{z}{p}) &= \frac{1}{\mathcal{N}}\sum_{l\ge0}(-1)^{l+1}q^{l(l+1)}\sum_{k>l}\frac{(-1)^kq^{k^2}z^{2k}}{(q)_{2k}}\\ &= \frac{1}{\mathcal{N}}\sum_{l\ge0}(-1)^{l+1}q^{l(l+1)}\bigg[\mathcal{N}-\sum_{0\le k\le l}\frac{(-1)^kq^{k^2}z^{2k}}{(q)_{2k}}\bigg]\\ &= \sum_{l\ge0}(-1)^{l+1}q^{l(l+1)}+\frac{1}{\mathcal{N}}\sum_{k\ge0}\frac{(-1)^kq^{k^2}z^{2k}}{(q)_{2k}}\sum_{l\ge k}(-1)^lq^{l(l+1)}.\end{aligned}$$

Again, we don't know how to simplify this any further[1], but get again, letting $q\to1$ in $S^{<>}(z)$ the expected result

$$-\frac12+\frac12\frac{\cosh z}{\cos z}.$$

Note also the form

$$S^{<>}(\tfrac{z}{p}) = \frac{1}{\mathcal{N}}\sum_{l\ge1}\frac{(-1)^lq^{l^2}z^{2l}}{(q)_{2l}}\sum_{0\le k<l}(-1)^{k+1}q^{k(k+1)}.$$

The first few values are

$$q,\ q^2(1+q+q^2+q^3+2q^4),\ q^3(1+2q+3q^2+5q^3+8q^4+9q^5+11q^6+10q^7+9q^8+7q^9+5q^{10}+3q^{11}+3q^{12}).$$

6. The instance $\ge<$

It is easier to switch left with right and thus start with an odd number of up-steps, followed by an arbitrary repetition of down-up's.

$$g(u) = \frac{p^2uz^2}{(qu)_2}+\frac{p^2z^2}{q^2(qu)_2}g(q^2u),$$

whence

$$g(u) = u\sum_{l\ge1}\frac{(pz)^{2l}}{(qu)_{2l}}.$$

And now

$$f(u) = g(u)+\frac{p^2uz^2}{(qu)_2}f(1)-\frac{p^2z^2}{q^2(qu)_2}f(q^2u).$$

[1]Compare the remark at the end of this paper.

Iterating and specializing:

$$f(1)=S^{\ge<}(z)=\sum_{k\ge0}\frac{(-1)^k g(q^{2k})(pz)^{2k}}{q^{2k}(q)_{2k}}\Bigg/\sum_{k\ge0}\frac{(-1)^k(pz)^{2k}}{(q)_{2k}}.$$

So, with

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^k z^{2k}}{(q)_{2k}},$$

$$\begin{aligned}S^{\ge<}(\tfrac{z}{p})&=\frac{1}{\mathcal{N}}\sum_{k\ge0}\frac{(-1)^k z^{2k}}{q^{2k}(q)_{2k}}q^{2k}\sum_{l\ge1}\frac{z^{2l}}{(q^{2k+1})_{2l}}\\&=\frac{1}{\mathcal{N}}\sum_{k\ge0}(-1)^k z^{2k}\sum_{l\ge1}\frac{z^{2l}}{(q)_{2k+2l}}=S^{\le>}(\tfrac{z}{p}).\end{aligned}$$

7. The instance $\ge\le$

With the same function $g(u)$ from the previous section,

$$f(u)=g(u)+\frac{p^2uz^2}{(qu)_2}f(1)-\frac{p^2uz^2}{(qu)_2}f(q^2u).$$

Iterating and specializing:

$$f(1)=S^{\ge\le}(z)=\sum_{k\ge0}\frac{(-1)^k g(q^{2k})(pz)^{2k}q^{k(k-1)}}{(q)_{2k}}\Bigg/\sum_{k\ge0}\frac{(-1)^k(pz)^{2k}q^{k(k-1)}}{(q)_{2k}}.$$

So, with

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^k z^{2k}q^{k(k-1)}}{(q)_{2k}},$$

$$\begin{aligned}S^{\ge\le}(\tfrac{z}{p})&=\frac{1}{\mathcal{N}}\sum_{k\ge0}\frac{(-1)^k z^{2k}q^{k(k-1)}}{q^{2k}(q)_{2k}}q^{2k}\sum_{l\ge1}\frac{z^{2l}}{(q^{2k+1})_{2l}}\\&=\frac{1}{\mathcal{N}}\sum_{k\ge0}(-1)^k z^{2k}q^{k(k-1)}\sum_{l\ge1}\frac{z^{2l}}{(q)_{2k+2l}}\\&=\frac{1}{\mathcal{N}}\sum_{l\ge1}\frac{z^{2l}}{(q)_{2l}}\sum_{0\le k<l}(-1)^k q^{k(k-1)}.\end{aligned}$$

The first few values are

$$1,\ (1+q^2)(1+q+q^2),\ (1+q^2)(1+2q+4q^2+4q^3+5q^4+6q^5+6q^6+4q^7+3q^8+2q^9+q^{10}).$$

8. The instance $><$

$$g(u)=\frac{p^2qu^2z^2}{(qu)_2}+\frac{p^2qu^2z^2}{(qu)_2}g(q^2u),$$

whence

$$g(u)=\sum_{l\ge1}\frac{(pz)^{2l}u^{2l}q^{2l^2-l}}{(qu)_{2l}}.$$

And now

$$f(u)=g(u)+\frac{p^2qu^2z^2}{(qu)_2}f(1)-\frac{p^2uz^2}{q(qu)_2}f(q^2u).$$

Iterating and specializing:

$$f(1)=S^{\ge<}(z)=\sum_{k\ge0}\frac{(-1)^kg(q^{2k})(pz)^{2k}q^{k^2-1}}{(q)_{2k}}\Big/\sum_{k\ge0}\frac{(-1)^k(pz)^{2k}q^{k^2}}{(q)_{2k}}.$$

So, with

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^kz^{2k}q^{k^2}}{(q)_{2k}},$$

$$\begin{aligned}S^{\ge<}(\tfrac{z}{p})&=\frac{1}{\mathcal{N}}\sum_{k\ge0}\frac{(-1)^kz^{2k}q^{k^2-1}}{(q)_{2k}}\sum_{l\ge1}\frac{z^{2l}q^{(2k+1)2l}q^{2l^2-l}}{(q^{2k+1})_{2l}}\\&=\frac{1}{\mathcal{N}}\sum_{k\ge0}\sum_{l\ge1}\frac{(-1)^kz^{2k+2l}q^{(2k+1)2l+2l^2-l+k^2-1}}{(q)_{2k+2l}}\\&=\frac{1}{\mathcal{N}}\sum_{l\ge1}\frac{z^{2l}q^{l(2l+1)}}{(q)_{2l}}\sum_{0\le k<l}(-1)^kq^{-1-k(k+1)}.\end{aligned}$$

The first few values are

$$q^2,\ q^3(1+q+2q^2+q^3+q^6),\ q^4(1+q^2)(1+2q+3q^2+4q^3+5q^4+5q^5+5q^6+4q^7+3q^8+2q^9+3q^{10}+q^{11}-q^{12}+q^{14}).$$

9. The instance $>\le$

We have the same function $g(u)$ as in the previous section, and the functional equation

$$f(u)=g(u)+\frac{p^2qu^2z^2}{(qu)_2}f(1)-\frac{p^2qu^2z^2}{(qu)_2}f(q^2u).$$

Iterating and specializing:

$$f(1)=S^{\ge<}(z)=\sum_{k\ge0}\frac{(-1)^kg(q^{2k})(pz)^{2k}q^{k(2k-1)}}{(q)_{2k}}\Big/\sum_{k\ge0}\frac{(-1)^k(pz)^{2k}q^{k(2k-1)}}{(q)_{2k}}.$$

So, with

$$\mathcal{N}:=\sum_{k\ge0}\frac{(-1)^kz^{2k}q^{k(2k-1)}}{(q)_{2k}},$$

$$\begin{aligned}
S^{\geq<}(\tfrac{z}{p}) &= \frac{1}{\mathcal{N}}\sum_{k\geq 0}\frac{(-1)^k z^{2k}q^{k(2k-1)}}{(q)_{2k}}\sum_{l\geq 1}\frac{z^{2l}q^{(2k+1)2l}q^{2l^2-l}}{(q^{2k+1})_{2l}}\\
&= \frac{1}{\mathcal{N}}\sum_{k\geq 0}\sum_{l\geq 1}\frac{(-1)^k z^{2k+2l}q^{(2k+1)2l+2l^2-l+2k^2-k}}{(q)_{2k+2l}}\\
&= \frac{1}{\mathcal{N}}\sum_{l\geq 1}\frac{z^{2l}q^{l(2l+1)}}{(q)_{2l}}\sum_{0\leq k<l}(-1)^k q^{-2k}\\
&= \frac{1}{\mathcal{N}}\sum_{l\geq 1}\frac{z^{2l}q^{l(2l+1)}}{(q)_{2l}}\frac{1-(-1)^l q^{-2l}}{1+q^{-2}}\\
&= \frac{1}{1+q^{-2}}\frac{1}{\mathcal{N}}\sum_{l\geq 0}\frac{z^{2l}q^{l(2l+1)}}{(q)_{2l}} - \frac{1}{1+q^{-2}}.
\end{aligned}$$

Here, the limiting form

$$\frac{\cosh z}{2\cos z} - \frac{1}{2}$$

is particularly obvious.

The first few values are

$$q^3,\ q^4(1+q+2q^2+q^3+q^6),\ q^5(1+q^2)(1+2q+4q^2+5q^3+5q^4+5q^5+6q^6+4q^7+q^8+q^9+3q^{10}+q^{11}-q^{12}+q^{14}).$$

10. Conclusion

Guo and Zeng might like to add some of the more attractive forms obtained in this paper to their list, like

$$\sum_{k\geq 0}\frac{z^{2k}q^{k(2k-1)}}{(q)_{2k}}\Big/\sum_{k\geq 0}\frac{(-1)^k z^{2k}q^{k(2k-1)}}{(q)_{2k}},$$
$$\sum_{k\geq 0}\frac{z^{2k}q^{k(2k+1)}}{(q)_{2k}}\Big/\sum_{k\geq 0}\frac{(-1)^k z^{2k}q^{k(2k-1)}}{(q)_{2k}}.$$

We leave the future study of q-Salié numbers introduced in this paper to further research and/or other researchers.

Remark. From Warnaar [4] we get, by specializing one of his beautiful formulæ:

$$\sum_{0\leq k\leq n}(-1)^k q^{\binom{k}{2}} = [z^n](q)_\infty(qz)_\infty\sum_{k\geq 0}\frac{q^k}{(q)_k(z)_k}.$$

Replacing q by q^2 "evaluates" the sums

$$\sum_{0\leq k\leq n}(-1)^k q^{k(k-1)}$$

that appear frequently in this paper.

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