PSEUDO q-ENGEL EXPANSIONS AND ROGERS-RAMANUJAN TYPE IDENTITIES

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Abstract. Andrews, Knopfmacher and Knopfmacher have used the Schur polynomials to consider the celebrated Rogers-Ramanujan identities in the context of q-Engel expansions. We extend this view using similar polynomials, provided by Sills, in the context of Slater's list of 130 Rogers-Ramanujan type identities.

1. Introduction

The idea to adapt the Engel expansion to formal power series is due to the late John Knopfmacher [?], as frequently mentioned by his son and coauthor (of [?] and other papers) Arnold Knopfmacher. The authors of this paper joined forces with George Andrews, the great specialist of partitions and q-series, and consequently the q-Engel expansion was established for the celebrated Rogers-Ramanujan identities [?]. When Peter Paule visited Johannesburg in 1999, his expertise played a pivotal role in the application of the Engel concept to the recently established m-generalisation of the Rogers-Ramanujan identities due to Garrett, Ismail, and Stanton [?], see [?]. Two more papers appeared a little later [?, ?] — and then the q-Engel expansion went into hibernation!

However, after all this, an important paper appeared: Drew Sills' annotated list [?] of Slater's list [?], in which he constructed polynomial recursions for *all* 130 identities. In the classical case, these polynomials are well known as *Schur polynomials* [?]. Since these polynomials are at the heart of all the q-Engel expansions, it is now an excellent opportunity to reawaken the subject by using Sills' polynomials. (Not all of them were new, and the so-called Santos polynomials appeared already in the papers described above.)

Now let us describe the subject with as little *jargon* as possible: A is a formal power series in the variable q, and we have the recursion $A_0 := A$, $A_{n+1} = a_nA_n - 1$. The "digits" a_n are polynomials in the variable q^{-1}, and the A_n are formal Laurent series. To a formal Laurent series $f = \sum_{n\ge\nu} c_nq^n$, define

$$[f] := \sum_{\nu\le n\le 0} c_nq^n.$$

If $a_0 = [A_0]$ and $a_n = [1/A_n]$ for all $n \ge 1$, then we have by formal iteration the identity

$$A = a_0 + \sum_{n\ge1} \frac{1}{a_1 \dots a_n}.$$

It is called the q-Engel expansion of A.

We use the traditional notation of q-analysis, as can be found in the book [?]: $(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$, $(x;q)_\infty := (1-x)(1-xq)\dots$, $(x,y,\dots,z;q)_\infty := (x;q)_\infty(y;q)_\infty\dots(z;q)_\infty$.

As a warm-up, let us review the essential steps related to the Rogers-Ramanujan identity

$$\sum_{n\ge0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}.$$

The polynomial recursion is $P_n = P_{n-1} + q^{n-1}P_{n-2}$, with $P_0 = P_1 = 1$ (A.18 in Sills' list [?]), and the limit $P_\infty = 1/(q,q^4;q^5)_\infty$. A point of clarification about this list is in order: Equation (18) from Slater's paper [?] appears in the annotated list as (A.18) for easy cross-referencing. The corresponding polynomial generalization appears in section 3 of [?] as Eq. (3.18).

We have $a_0 = 1$, and set

$$X_n := \sum_{k\ge0} q^{kn}P_k.$$

Then, by summation,

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n} P_{k-1} + \sum_{k=1}^{n} q^{k-1}P_{k-2},$$

and going to the limit,

$$P_\infty = 1 + \sum_{k\ge2} q^{k-1}P_{k-2} = 1 + qX_1.$$

Furthermore,

$$\begin{aligned}X_n &= 1 + q^n + \sum_{k\ge2} q^{kn}P_k\\ &= 1 + q^n + \sum_{k\ge2} q^{kn}[P_{k-1} + q^{k-1}P_{k-2}]\\ &= 1 + q^n + q^n\sum_{k\ge0} q^{kn}P_k - q^n + q^{2n+1}\sum_{k\ge0} q^{k(n+1)}P_k\\ &= 1 + q^nX_n + q^{2n+1}X_{n+1},\end{aligned}$$

or

$$X_n = \frac{1}{1-q^n} + \frac{q^{2n+1}}{1-q^n}X_{n+1}.$$

By iteration,

$$P_\infty = 1 + q\left[\frac{1}{1-q} + \frac{q^3}{1-q}\left[\frac{1}{1-q^2} + \frac{q^5}{1-q^3}\left[\dots\right]\right]\right] = \sum_{n\ge0}\frac{q^{n^2}}{(q;q)_n}.$$

The X_n can easily be translated into A_n, and the digits can be pulled out.

Now, as we will see from the examples that we looked at, the condition that $a_n = [1/A_n]$ is hardly ever satisfied. Yet, all hope is not lost: What happens is for instance that $[a_n] = [1/A_n]$, since a_n might have positive powers of q. Or, at least, a_n and $1/A_n$ coincide for $n = \nu$ and a few more exponents and then don't agree anymore.

What is the problem here? One can still formally iterate and get the lefthand-side (the sum-side) of a Rogers-Ramanujan type identity. We lose uniqueness, and cannot compute the digits anymore. But the digits are *known*! And in all cases that we investigated, we could do essentially the same as what we just explained on the classical example. We thus call such an expansion that follows from Sills' polynomials a *pseudo q-Engel expansion.*

The is also a mild modification of the q-Engel machinery, which is basically a shift, so that one can pull out more coefficients, say

$$\sum_{\nu\le n\le 2} c_n q^n.$$

This would certainly not help in the case that a_n is a Laurent series with arbitrarily high powers. Also, the iteration (here described in terms of X_n and X_{n+1}) is what it is and cannot be changed. However, the polynomials P_n with the appropriate limit P_∞ are *not* unique, and it could very well be that another such sequence Q_n would lead to a proper q-Engel expansion instead of just a pseudo q-Engel expansion. However, we worked on Sills' list, and will report this now in the following sections. As the reader will notice, the scheme is always very similar indeed, and thus we stopped after a while. It would be a good project for a student to work this out for *all* identities.

2. The first Göllnitz-Gordon identity (A.36)

This identity is

$$\sum_{n\ge 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty}.$$

We know from Sills' list that it equals the limit P_∞ of the recursion

$$P_n = (1+q^{2n-1})P_{n-1} + q^{2n-2}P_{n-2}, \quad P_0 = 1,\ P_1 = 1+q.$$

Regarding the sum side as a candidate for a q-Engel expansion, we get the digits

$$a_n = \frac{1-q^{2n}}{q^{2n-1}(1+q^{2n-1})}, \quad n \ge 1.$$

We see that a_n has negative *and positive* powers in its Laurent expansion. Thus, this *cannot* be a q-Engel expansion. However, we will see that it is a pseudo q-Engel expansion, i. e., the nonpositive powers of a_n are as they should be.

We sum the recursion between 1 and n:

$$\sum_{k=1}^n P_k = \sum_{k=1}^n P_{k-1} + \sum_{k=1}^n q^{2k-1}P_{k-1} + \sum_{k=1}^n q^{2k-2}P_{k-2},$$

or

$$P_n = 1 + \sum_{k=1}^n q^{2k-1}P_{k-1} + \sum_{k=2}^n q^{2k-2}P_{k-2}.$$

Taking the limit $n \to \infty$,

$$P_\infty = 1 + q(1+q)\sum_{k\ge 0} q^{2k}P_k = 1 + q(1+q)X_1,$$

with

$$\begin{aligned} X_n &:= \sum_{k\ge 0} q^{2kn}P_k \\ &= \sum_{k\ge 2} q^{2kn}P_k + 1 + (1+q)q^{2n} \\ &= \sum_{k\ge 2} q^{2kn}[(1+q^{2k-1})P_{k-1} + q^{2k-2}P_{k-2}] + 1 + (1+q)q^{2n} \\ &= \sum_{k\ge 0} q^{2kn+2n}(1+q^{2k+1})P_k - q^{2n}(1+q) + \sum_{k\ge 0} q^{2kn+4n}q^{2k+2}P_k + 1 + (1+q)q^{2n} \\ &= q^{2n}X_n + q^{2n+1}X_{n+1} + q^{4n+2}X_{n+1} + 1. \end{aligned}$$

Hence

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{2n+1}(1+q^{2n+1})}{1-q^{2n}}X_{n+1},$$

and

$$\begin{aligned} P_\infty &= 1 + q(1+q)X_1 \\ &= 1 + q(1+q)\left[\frac{1}{1-q^2} + \frac{q^3(1+q^3)}{1-q^2}X_2\right] \\ &= 1 + q(1+q)\left[\frac{1}{1-q^2} + \frac{q^3(1+q^3)}{1-q^2}\left[\frac{1}{1-q^4} + \frac{q^5(1+q^5)}{1-q^4}X_3\right]\right] \\ &= 1 + \frac{q(1+q)}{(q^2;q^2)_1} + \frac{q^4(1+q)(1+q^3)}{(q^2;q^2)_2} + \cdots \\ &= \sum_{n\ge 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}. \end{aligned}$$

Now we establish the connection between X_n and A_n:

$$(1-q^{2n})X_n = 1 + \frac{1}{a_{n+1}}(1-q^{2n+2})X_{n+1},$$

or

$$a_{n+1}[(1-q^{2n})X_n - 1] - 1 = (1-q^{2n+2})X_{n+1} - 1,$$

so that we have to set

$$\begin{aligned} A_n &:= (1-q^{2(n-1)})X_{n-1} - 1 \\ &= \sum_{k\ge 1} q^{2k(n-1)}(P_k - P_{k-1}) \\ &= q^{2(n-1)}q + q^{4(n-1)}(q^2+q^3+q^4) + O(q^{6(n-1)+4}) \\ &= q^{2n-1}\big(1 + q^{2n-1}(1+q^2+q^3) + O(q^{4n-1})\big), \end{aligned}$$

and

$$\frac{1}{A_n} = \frac{1}{q^{2n-1}}\big(1 - q^{2n-1}(1+q^2+q^3) + O(q^{4n-2})\big) = \frac{1}{q^{2n-1}} + O(1) = a_n + O(1).$$

The second Göllnitz-Gordon identity (A.34). This identity is

$$\sum_{n\geq 0} \frac{q^{n(n+2)}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}.$$

We know from Sills' list that it equals the limit P_∞ of the recursion

$$P_n = (1+q^{2n+1})P_{n-1} + q^{2n}P_{n-2}, \quad P_0 = 1,\ P_1 = 1+q^3.$$

A very similar computation gives

$$P_\infty = 1 + q^3(1+q)X_1$$

with

$$X_n = q^{2n}X_n + q^{2n+3}X_{n+1} + q^{4n+4}X_{n+1} + 1.$$

Hence

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{2n+3}(1+q^{2n+1})}{1-q^{2n}}X_{n+1}$$

and

$$\begin{aligned} P_\infty &= 1 + q^3(1+q)X_1 \\ &= 1 + q^3(1+q)\left[\frac{1}{1-q^2} + \frac{q^5(1+q^3)}{1-q^2}X_2\right] \\ &= \sum_{n\geq 0} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}. \end{aligned}$$

Again we have

$$a_n = \frac{1-q^{2n}}{q^{2n+1}(1+q^{2n-1})}, \quad n \geq 1$$

and

$$A_n = \sum_{k\geq 1} q^{2k(n-1)}(P_k - P_{k-1}) = q^{2n+1} + q^{4n}(1+q+q^4) + \cdots$$

and thus

$$\frac{1}{A_n} = \frac{1}{q^{2n+1}}\big(1 - q^{2n-1}(1+q)\big) + O(q) = a_n + O(q).$$

3. A.20 and A.16

The first identity is

$$\sum_{n\ge0}\frac{q^{n^2}}{(q^4;q^4)_n}=\frac{1}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty}.$$

Here, the digits are given as

$$a_n=\frac{1-q^{4n}}{q^{2n-1}}.$$

Again, this cannot be a q-Engel expansion, since there are negative and positive powers. The recursion from Sills' paper is

$$P_n=(1-q^2+q^{2n-1})P_{n-1}+q^2P_{n-2},\quad P_0=1,\ P_1=1+q.$$

The usual computation gives

$$P_\infty=1+\frac{q}{1+q^2}X_1,$$

and

$$X_n=\frac{1}{1-q^{2n}}+\frac{q^{2n+1}}{(1-q^{2n})(1+q^{2n+2})}X_{n+1},$$

from which the desired form for P_∞ follows by iteration. Again,

$$A_n=\sum_{k\ge1}q^{2k(n-1)}(P_k-P_{k-1})=q^{2(n-1)}q+q^{4(n-1)}q^4+\cdots,$$

and

$$\frac{1}{A_n}=\frac{1}{q^{2n-1}}+O(q^2)=a_n+O(q^{4n}).$$

Now we move to A.16:

$$\sum_{n\ge0}\frac{q^{n(n+2)}}{(q^4;q^4)_n}=\frac{1}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty}.$$

Here, the digits are given as

$$a_n=\frac{1-q^{4n}}{q^{2n+1}}.$$

The recursion from Sills' paper is

$$P_n=(1-q^2+q^{2n+1})P_{n-1}+q^2P_{n-2},\quad P_0=1,\ P_1=1+q^3.$$

We get

$$P_\infty=1+\frac{q^3}{1+q^2}X_1$$

with

$$X_n=\frac{1}{1-q^{2n}}+\frac{q^{2n+3}}{(1-q^{2n})(1+q^{2n+2})}X_{n+1};$$

$$P_\infty=1+\frac{q^3}{1+q^2}X_1=\sum_{n\ge0}\frac{q^{n(n+2)}}{(q^4;q^4)_n}.$$

Again,

$$A_n = \sum_{k\ge 1} q^{2k(n-1)}(P_k - P_{k-1}) = q^{2n+1}(1 + q^{2n+3} + O(q^{4n}))$$

and

$$\frac{1}{A_n} = \frac{1}{q^{2n+1}} + O(q^2) = a_n + O(q^{2n-1}).$$

4. A.79 and A.99

This is

$$\sum_{n\ge 0} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{(q^8, q^{12}; q^{20})_\infty (-q;q)_\infty}{(q;q)_\infty}.$$

Digits:

$$a_n = \frac{(1-q^{2n-1})(1-q^{2n})}{q^{2n-1}} = \frac{1}{q^{2n-1}} - 1 + \cdots$$

Recursion:

$$P_n = (1 + q + q^{2n-1})P_{n-1} - qP_{n-2}, \quad P_0 = 1,\ P_1 = 1 + q.$$

Limit:

$$P_\infty = 1 + \frac{q}{1-q} \sum_{k\ge 0} X_1,$$

with

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{2n+1}}{(1-q^{2n})(1-q^{2n+1})} X_{n+1},$$

from which the expression for P_∞ can be obtained by iteration. And

$$A_n = \sum_{k\ge 1} q^{2k(n-1)}(P_k - P_{k-1}) = q^{2n-1}\big(1 + q^{2n-1} + O(q^{2n-1})\big).$$

Thus

$$\frac{1}{A_n} = \frac{1}{q^{2n-1}} - 1 + O(q).$$

Identity A.99 is

$$\sum_{n\ge 0} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \frac{(q, q^9, q^{10}; q^{10})_\infty (q^8, q^{12}; q^{20})_\infty}{(q;q)_\infty}.$$

Digits:

$$a_n = \frac{(1-q^{2n-1})(1-q^{2n})}{q^{2n}} = \frac{1}{q^{2n}} - \frac{1}{q} + \cdots$$

Recursion:

$$P_n = (1 + q + q^{2n})P_{n-1} - qP_{n-2}, \quad P_0 = 1,\ P_1 = 1 + q^2.$$

Limit:

$$P_\infty = 1 + \frac{q^2}{1-q} X_1,$$

with

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{2n+2}}{(1-q^{2n})(1-q^{2n+1})} X_{n+1}$$

Also

$$A_n = \sum_{k\ge1} q^{2k(n-1)}(P_k - P_{k-1}) = q^{2n} + q^{4n-1} + \cdots$$

and

$$\frac{1}{A_n} = \frac{1}{q^{2n}} - \frac{1}{q} + \cdots$$

5. A.39 and A.38

These identities have been treated already in the literature, although with different polynomials. Let us quickly browse through them:

A.39:

$$P_n = (1+q)P_{n-1} + (q^{2n-2} - q)P_{n-2}, \quad P_0 = P_1 = 1.$$

$$P_\infty = 1 + \frac{q^2}{1-q} X_1,$$

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{4n+2}}{(1-q^{2n})(1-q^{2n+1})} X_{n+1},$$

$$\begin{aligned} P_\infty &= 1 + \frac{q^2}{1-q} X_1 \\ &= 1 + \frac{q^2}{1-q}\left[\frac{1}{1-q^2} + \frac{q^6}{(1-q^2)(1-q^3)} X_2\right] \\ &= 1 + \frac{q^2}{1-q}\left[\frac{1}{1-q^2} + \frac{q^6}{(1-q^2)(1-q^3)}\left[\frac{1}{1-q^4} + \frac{q^8}{(1-q^4)(1-q^5)} X_3\right]\right] \\ &= \sum_{n\ge0} \frac{q^{2n^2}}{(q;q)_{2n}}, \end{aligned}$$

$$a_n = \frac{(1-q^{2n-1})(1-q^{2n})}{q^{4n-2}} = \frac{1}{q^{4n-2}} - \frac{1}{q^{2n-1}} - \frac{1}{q^{2n-2}} + O(q),$$

$$A_n = \sum_{k\ge1} q^{2k(n-1)}(P_k - P_{k-1}) = q^{4n-2} + q^{6n-3}(1+q) + q^{8n-4},$$

$$\frac{1}{A_n} = \frac{1}{q^{4n-2}} - \frac{1}{q^{2n-1}} - \frac{1}{q^{2n-2}} + O(1),$$

so this is "almost" a q-Engel expansion, but the constant term differs.

A.38

This is

$$\sum_{n\ge0} \frac{q^{2n^2+2n}}{(q^2;q)_{2n+1}} = \frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q)_\infty}.$$

The recursion is

$$P_n = (1+q)P_{n-1} + (q^{2n} - q)P_{n-2}, \quad P_0 = 1,\ P_1 = 1+q.$$

We like the term for $n = 0$ to be 1. Thus we multiply by $1 - q$ and will then have the appropriate limit. This means just a change of initial values:

$$P_n = (1+q)P_{n-1} + (q^{2n} - q)P_{n-2}, \quad P_0 = 1-q,\ P_1 = 1-q^2,$$

Limit:

$$P_\infty = 1 + \frac{q^4}{1-q}X_1,$$

$$X_n = \frac{1-q}{(1-q^{2n})(1-q^{2n+1})} + \frac{q^{4n+4}}{(1-q^{2n})(1-q^{2n+1})}X_{n+1};$$

$$P_\infty = 1 + \frac{q^4}{1-q}X_1 = \sum_{n\ge 0} \frac{q^{2n^2+2n}}{(q^2;q)_{2n}}.$$

Digits:

$$a_n = \frac{(1-q^{2n})(1-q^{2n+1})}{q^{4n}} = \frac{1}{q^{4n}} - \frac{1}{q^{2n}} - \frac{1}{q^{2n-1}} + \cdots;$$

$$(1-q^{2n})(1-q^{2n+1})X_n = 1 - q + \frac{1}{a_{n+1}}(1-q^{2n+2})(1-q^{2n+3})X_{n+1},$$

and therefore

$$A_n = \frac{(1-q^{2n-2})(1-q^{2n-1})}{1-q}X_{n-1} - 1 = q^{4n}(1 + (1+q)q^{2n} + O(q^{3n-8})),$$

$$\frac{1}{A_n} = \frac{1}{q^{4n}} - \frac{1}{q^{2n}} - \frac{1}{q^{2n-1}} + \cdots.$$

6. Recursions of third order: the second Rogers-Selberg identity (A.32)

The identity in question is

$$\sum_{n\ge 0} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q^2)_{2n}} = \frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}.$$

Here, the digits are

$$a_n = \frac{(1-q^{2n})(1+q^{2n-1})(1+q^{2n})}{q^{4n}} = \frac{1}{q^{4n}} + \frac{1}{q^{2n+1}} - 1 + \cdots$$

and this has positive and negative powers, and cannot be a proper q-Engel expansion. The recursion from Sills' paper is

$$P_n = (1-q-q^2)P_{n-1} + (q^{2n} - q^3 + q^2 + q)P_{n-2} + q^3P_{n-3},\ P_0 = P_1 = 1, P_2 = 1+q^4.$$

A similar procedure as before leads to

$$P_\infty = 1 + \frac{q^4}{(1+q)(1+q^2)}X_1,$$

and

$$X_n = \frac{1}{1-q^{2n}} + \frac{q^{4n+4}}{(1-q^{2n})(1+q^{2n+1})(1+q^{2n+2})} X_{n+1},$$

from which the desired form follows from iteration. Again,

$$\begin{aligned} A_n &:= (1-q^{2n-2})X_{n-1} - 1 \\ &= \sum_{k\ge 1} q^{2k(n-1)}(P_k - P_{k-1}) \\ &= q^{4n}(1 - q^{2n-1} + q^{4n-2}(1+q^2+q^6) + \cdots) \end{aligned}$$

and

$$\frac{1}{A_n} = \frac{1}{q^{4n}} + \frac{1}{q^{2n+1}} - 1 + \cdots$$

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