A MATRIX WITH SUMS OF CATALAN NUMBERS—LU-DECOMPOSITION AND DETERMINANT

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Abstract. Following Benjamin et al., a matrix with entries being sums of two neighbouring Catalan numbers is considered. Its LU-decomposition is given, by guessing the results and later prove it by computer algebra, with lots of human help. Specializing a parameter, the determinant turns out to be a Fibonacci number with odd index, confirming earlier results, obtained back then by combinatorial methods.

1. Introduction

Let $\mathscr{C}_n = \frac{1}{n+1}\binom{2n}{n}$ be the n-th Catalan number. The $n \times n$ Matrix

$$\mathscr{M} = \begin{pmatrix} \mathscr{C}_t + \mathscr{C}_{t+1} & \mathscr{C}_{t+1} + \mathscr{C}_{t+2} & \dots & \mathscr{C}_{t+n-1} + \mathscr{C}_{t+n} \\ \mathscr{C}_{t+1} + \mathscr{C}_{t+2} & \mathscr{C}_{t+2} + \mathscr{C}_{t+3} & \dots & \mathscr{C}_{t+n} + \mathscr{C}_{t+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{C}_{t+n-1} + \mathscr{C}_{t+n} & \mathscr{C}_{t+n} + \mathscr{C}_{t+n+1} & \dots & \mathscr{C}_{t+2n-2} + \mathscr{C}_{t+2n-1} \end{pmatrix}$$

is considered in [1]; the determinant is considered by combinatorial means. The natural range of the parameters is $n \geq 1$ and $t \geq 0$. There are many methods to compute determinants of combinatorial matrices, as expertly described in [2, 3].

In this paper, we consider the LU-decomposition $LU = \mathscr{M}$, with a lower triangular matrix L with 1's on the main diagonal, and an upper triangular matrix U. From this, the determinant comes out as a corollary, by multiplying the elements in U's main diagonal. We restrict our attention to the instance $t = 0$, since the computations seem to become very messy in the more general setting. But at the same time, we consider a more general matrix with an extra parameter x, viz.

$$\mathscr{M} = \begin{pmatrix} \mathscr{C}_0 + x\mathscr{C}_1 & \mathscr{C}_1 + x\mathscr{C}_2 & \dots & \mathscr{C}_{n-1} + x\mathscr{C}_n \\ \mathscr{C}_1 + x\mathscr{C}_2 & \mathscr{C}_2 + x\mathscr{C}_3 & \dots & \mathscr{C}_n + x\mathscr{C}_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{C}_{n-1} + x\mathscr{C}_n & \mathscr{C}_n + x\mathscr{C}_{n+1} & \dots & \mathscr{C}_{2n-2} + x\mathscr{C}_{2n-1} \end{pmatrix}.$$

Not only do we get more general results in this way, but it is actually easier to guess the explicit forms of L and U with an extra parameter involved.

2010 *Mathematics Subject Classification.* 05A15, 11B39, 15B36 .

Key words and phrases. Catalan numbers, LU-decomposition, determinant, computer algebra.

Here are the results that we found by computer experiments, which we consider to be the main contributions of this paper:

Theorem 1. *For $k,i\ge 1$, set*

$$F(k,i)=\frac{1}{i(2i-1)}\binom{2i}{i-k}\sum_{0\le r\le k}\frac{1}{2k-r}\binom{2k-r}{r}\Big(ri+2ik^2-ik-2rk^2+2k^3-k^2\Big)x^r$$

and

$$g(k)=\sum_{0\le r\le k}\binom{2k-r}{r}x^r=F(k,k).$$

Then

$$L[i,k]=\frac{F(k,i)}{g(k)}\quad \textit{and}\quad U[k,j]=\frac{F(k,j)}{g(k-1)}.$$

In the next section, first the expressions for $F(i,j)$ and $g(k)$ will be simplified, and then it will be proved that these two matrices are indeed the LU-decomposition of $\mathscr{M}$. Note that only *one* function $F(k,i)$ is used to represent both, $L[i,k]$ and $U[k,j]$. This shows in particular the symmetry related to $i\leftrightarrow j$.

2. Simplification and proof

In many instances where Catalan numbers are involved, it is beneficial to work with an auxiliary variable:

$$x=\frac{-u}{(1+u)^2}\quad\text{and}\quad u=\frac{-1-2x+\sqrt{1+4x}}{2x}.$$

Then

$$g(k)=\frac{1-u^{2k+1}}{(1-u)(1+u)^{2k}}.$$

This is well within the reach of modern computer algebra (I use Maple). Further,

$$F(k,j)=(1-u^{2k})\frac{\binom{2j}{j-k}}{2j(2j-1)}\frac{2k^2-j}{(1-u)(1+u)^{2k-1}}+(1+u^{2k})\frac{\binom{2j}{j-k}k}{2j(1+u)^{2k}}.$$

Maple is capable to simplify $F(k,j)$, but the version given here, which is pleasant, was obtained with help from Carsten Schneider and his software [5]. Of course, once this version is known, Maple can confirm that it is equivalent to its own simplification. Note that $F(k,k)=g(k)$, and the L-matrix has indeed 1's on the main diagonal.

What is nice to note is that $L[i,k]=0$ for $i<k$ and $U[k,j]=0$ for $k>j$ automatically, thanks to the properties of binomial coefficients: a binomial coefficient $\binom{n}{m}$ with integers n,m such that $n\ge 0$ and $m<0$ is equal to zero.

Now we want to evaluate the (i,j) entry of the matrix $L\cdot U$:

$$\sum_{k\ge 1}L[i,k]U[k,j].$$

Maple cannot evaluate this sum without help:

$$\frac{F(k,i)F(k,j)}{g(k)g(k-1)} = \frac{\text{expression}}{(1-u^{2k+1})(1-u^{2k-1})}$$

What helps here is partial fraction decomposition:

$$\frac{F(k,i)F(k,j)}{g(k)g(k-1)} = \text{expression}_1 + \frac{\text{expression}_2}{(1-u^{2k+1})} + \frac{\text{expression}_3}{(1-u^{2k-1})}.$$

In the second term the change of index $k \to k-1$ makes things better, so that Maple can compute the sum over k; however, a correction term needs to be taken in:

$$\sum_{k=1}^{j} \frac{F(k,i)F(k,j)}{g(k)g(k-1)} = \sum_{k=1}^{j} \frac{\text{expression}_4}{(1-u^{2k-1})} - \frac{\text{expression}_2}{(1-u^{2k+1})}\Big|_{k=0}.$$

All the expressions are long and can be created with a computer. The sum can now be computed, and, switching back to the x-world, simplifies (again with a lot of human help, e. g., to simplify expressions in which the Gamma-functions appears) the last sum to

$$\mathscr{C}_{i+j-2} + x\mathscr{C}_{i+j-1},$$

as it should. For our simplification, we still used the variable u in [4]. However, for small x and u, the connection between the two variables is bijective.

All the details can be checked in the maple worksheet [4]. Perhaps a quick comment how the partial fraction decomposition is working is the essential formula

$$\frac{1}{g(k)g(k-1)} = (1-u)(1+u)^{4k-3}\Big[\frac{1}{1-u^{2k-1}} - \frac{u^2}{1-u^{2k+1}}\Big].$$

3. The determinant

The values in the main diagonal are given by

$$U[k,k] = \frac{g(k)}{g(k-1)}.$$

Consequently

$$\prod_{k=1}^{n} U[k,k] = \frac{g(n)}{g(0)} = g(n).$$

Setting $x=1$, as in [1], means $u = -\frac{3+\sqrt{5}}{2} = -\alpha^2$, with $\alpha = \frac{1+\sqrt{5}}{2}$ being the golden ratio. We also need $\beta = \frac{1-\sqrt{5}}{2}$. After some straightforward simplifications, this can be rewritten in terms of Fibonacci numbers:

$$g(n) = \frac{1+\alpha^{4n+2}}{(1-\alpha^2)^{2n}(1+\alpha^2)} = \frac{1+\alpha^{4n+2}}{\alpha^{2n}\sqrt{5}\alpha} = \frac{\alpha^{2n+1}-\beta^{2n+1}}{\sqrt{5}} = F_{2n+1}.$$

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