# KNÖDEL WALKS IN A BÖHM-HORNIK ENVIRONMENT

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Abstract. Ideas of Knödel and Böhm-Hornik about walks in certain graphs, resembling the classical symmetric random walk on the integers, are combined. All the relevant generating functions (although occasionally quite involved) are made fully explicit.

## 1. Introduction

The standard random walk on the non-negative integers may be visualized by the following graph (only the first 8 states are shown):

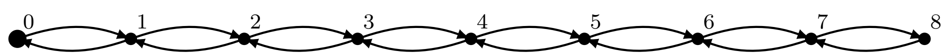


Figure 1. Standard symmetric random walk on the non-negative integers

One starts in state 0 and can go up/down one step, each with the same probability.

Böhm and Hornik introduces a related model: up-steps occur with probability $\alpha$ and down-steps occur with probability $\beta = 1-\alpha$, but after each step $\alpha$ and $\beta$ change their roles. The follow graph is useful to grasp the idea.

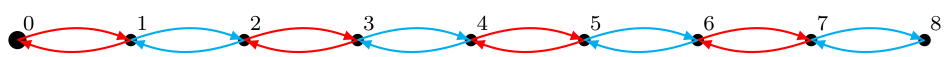


Figure 2. Red edges are labelled with the weight $\alpha$, blue edges with $\beta$

Böhm and Hornik [1] do not restrict the random walks to the non-negative integers. Alternative/additional analysis can be found in [3].

Another twist of a random walk occurs in a model introduced by Knödel [2]: There are bins of size 1 and small items (size $\frac{1}{3}$) and large items (size $\frac{2}{3}$) arrive with the same probability. States correspond to boxed filled with just one large item each. There is one exception, when a small item arrives at the origin. In this case, it cannot be used to complete a partially filled bin, and an extra state is introduced. See [4] and some referenced papers for analysis.

It is the purpose of this paper to combine the ideas of Knödel and Böhm-Hornik: Large items arrive with probability $\alpha$ and small items with probability $\beta$, but after each step the roles of $\alpha$ and $\beta$ are changed. The graph with two layers of states will explain the scenario readily. The rest of the paper is devoted to derive generating functions for walks starting at the origin and ending in a prescribed state. The kernel method [4] and the heavy use of computer algebra (Maple) will be essential.

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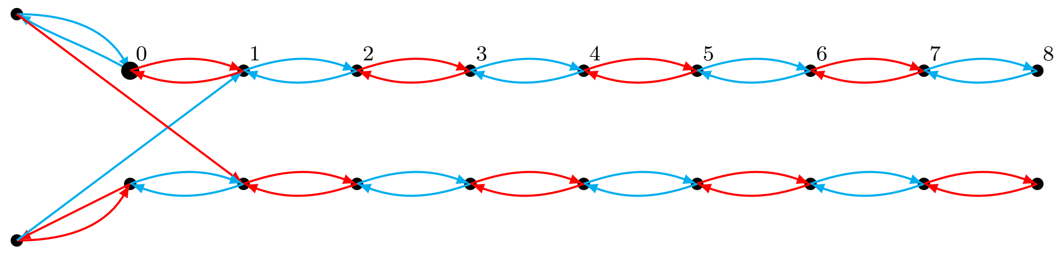


FIGURE 3. The Knödel-Böhm-Hornik graph

First, we start with a direct approach, which is a brute-force procedure. It leads to four equations, and eventually to biquadratic equations. Computers are capable of handling this, but the next section mostly serves as an invitation to a more sophisticated approach, using only two functions (not four). And, lo and behold, after a certain substitution, the ugly beast turns into a beautiful swan.

## 2. Brute-force Analysis

We introduce the following generating functions: $f_i = f_i(z)$ has as coefficient of $z^n$ the probability to reach state $i$ from the upper layer in $n$ steps, starting from the origin (state 0). The function $g_i$ is similar, but refers to the lower layer of states. Finally, the extra states and their generating functions are called $P$ resp. $Q$.

From the diagram, considering the last step made, one can see the recursions

$$\begin{aligned}
&f_i = \beta z f_{i-1} + \alpha z f_{i+1}, \ i = 2, 4, 6, \dots, \\
&f_i = \alpha z f_{i-1} + \beta z f_{i+1}, \ i = 3, 5, 7, \dots, \\
&f_1 = \alpha z f_0 + \beta z f_2 + \beta z Q = \alpha z f_0 + \beta z f_2 + \beta\alpha z^2 g_0, \\
&f_0 = 1 + \beta z P + \alpha z f_1 = 1 + \beta^2 z^2 f_0 + \alpha z f_1, \\
&P = \beta z f_0, \\
&g_i = \alpha z g_{i-1} + \beta z g_{i+1}, \ i = 2, 4, 6, \dots, \\
&g_i = \beta z g_{i-1} + \alpha z g_{i+1}, \ i = 3, 5, 7, \dots, \\
&g_1 = \beta z g_0 + \alpha z g_2 + \alpha z P = \beta z g_0 + \alpha z g_2 + \alpha\beta z^2 f_0, \\
&g_0 = \alpha z Q + \beta z g_1 = \alpha^2 z^2 g_0 + \beta z g_1, \\
&Q = \alpha z g_0.
\end{aligned}$$

In order to attack this system, we introduce a second variable $u$ and consider the following four bivariate generating functions:

$$F_e(u) = \sum_{i\ge0} u^{2i} f_{2i}, \quad F_o(u) = \sum_{i\ge0} u^{2i+1} f_{2i+1},$$
$$G_e(u) = \sum_{i\ge0} u^{2i} g_{2i}, \quad G_o(u) = \sum_{i\ge0} u^{2i+1} g_{2i+1};$$

'e' stands for 'even', 'o' stands for odd. Summing the first recursion, we find (omitting the variable $u$ for the moment)

$$F_e - f_0 = \beta z u F_o + \frac{\alpha z}{u}(F_o - u f_1);$$

adding the recursion for $f_0$ leads to

$$F_e = \beta zuF_o + \frac{\alpha z}{u}F_o + 1 + \beta^2 z^2 f_0.$$

Similarly, for the odd incides

$$F_o - uf_1 = \alpha zu(F_e - f_0) + \frac{\beta z}{u}(F_e - f_0 - u^2 f_2)$$

and further

$$F_o = \alpha zuF_e + \frac{\beta z}{u}(F_e - f_0) + u\beta\alpha z^2 g_0.$$

The same procedure is done for the even indices and the $g_i$'s:

$$G_e - g_0 = \alpha zuG_o + \frac{\beta z}{u}(G_o - ug_1)$$

and

$$G_e = \alpha zuG_o + \frac{\beta z}{u}G_o + \alpha^2 z^2 g_0.$$

Finally, for the odd indices

$$G_o - ug_1 = \beta zu(G_e - g_0) + \frac{\alpha z}{u}(G_e - g_0 - u^2 g_2)$$

and

$$G_o = \beta zuG_e + \frac{\alpha z}{u}(G_e - g_0) + u\alpha\beta z^2 f_0.$$

For the reader's convenience we collected the four equations that we (and Maple) have to deal with:

$$\begin{aligned}
F_e &= \beta zuF_o + \frac{\alpha z}{u}F_o + 1 + \beta^2 z^2 f_0,\\
F_o &= \alpha zuF_e + \frac{\beta z}{u}(F_e - f_0) + u\beta\alpha z^2 g_0,\\
G_e &= \alpha zuG_o + \frac{\beta z}{u}G_o + \alpha^2 z^2 g_0,\\
G_o &= \beta zuG_e + \frac{\alpha z}{u}(G_e - g_0) + u\alpha\beta z^2 f_0;
\end{aligned}$$

we note again that $f_0 = F_e(0)$ and $g_0 = G_e(0)$.

Maple can solve this, but the solution is implicit since it still depends on $f_0$ and $g_0$. The expressions are quite long, and they all share the same denominator $N$:

$$N = u^2 - z^2u^4\alpha - z^2u^2 + 2u^2\alpha z^2 + \alpha^2 z^2 u^4 - 2u^2\alpha^2 z^2 - z^2\alpha + z^2\alpha^2.$$

Then $NF_e = u^2 - \alpha f_0 z^2 + \alpha^2 f_0 z^2 - 2z^3u^4\alpha^2 g_0 + z^3u^4\alpha g_0 + z^3u^2\alpha^2 g_0 + z^3\alpha^3u^4 g_0 - z^3u^2\alpha^3 g_0$ and $NF_o = uz(-u^2\alpha^2 z g_0 - f_0 + \alpha f_0 + u^2\alpha z g_0 - z^2\alpha^3 f_0 + \alpha u^2 - \alpha + \alpha^3 z^2 u^2 f_0 + 1 + u^2\alpha z^2 f_0 - 2u^2\alpha^2 z^2 f_0 + f_0 z^2 - 3\alpha f_0 z^2 + 3\alpha^2 f_0 z^2)$ and $NG_e = -\alpha z^2(-\alpha u^4 z f_0 + \alpha^2 u^4 z f_0 + g_0 - u^2 z f_0 + 2u^2\alpha z f_0 - \alpha g_0 - u^2\alpha^2 z f_0)$ and finally $NG_o = -uz\alpha(-u^2\alpha z^2 g_0 + g_0 - u^2 z f_0 + u^2\alpha z f_0 + u^2\alpha^2 z^2 g_0 - \alpha^2 z^2 g_0)$.

The denominator $N$ has 4 roots, considering $u$ as the variable:

$$s_1 = \frac{\sqrt{\alpha(1-\alpha)(1 - 2z^2\alpha^2 + 2z^2\alpha - z^2 - \sqrt{(1-z)(1+z)(1-z+2z\alpha)(1+z-2z\alpha)})}}{\sqrt{2}z\alpha(1-\alpha)},$$

$$s_2 = -s_1,\ s_3 = \frac{1}{s_1},\ s_4 = \frac{1}{s_2}.$$

The factors $u - s_1$ and $u - s_2$ are 'bad' in the sense of the kernel method [4], i. e., they don't lead to a power series expansion around the origin. Consequently, the numerators of the four functions must be divisible by both factors. Applying this principle to $F_e$ and $G_e$ leads to two equations, from which $f_0$ and $g_0$ can be computed. Again, the expressions are long, and an auxiliary quantity $W$ is used:

$$W = \sqrt{(1-z)(z+1)(1-z+2z\alpha)(1+z-2z\alpha)}.$$

Here are the results:

$$f_0 = \frac{\Xi_1}{4\alpha^2 z^4(-1+z)(z+1)(-1+\alpha)^2(-1+z^2-3z^2\alpha+3z^2\alpha^2)}$$

with $\Xi_1 = (-3z^4\alpha^2 + \alpha^2 W z^2 + 3z^2\alpha^2 + 3z^4\alpha - aWz^2 - 3z^2\alpha - z^4 + Wz^2 + 2z^2 - 1 - W)(2z^2\alpha^2 - 2z^2\alpha + z^2 - 1 + W)$
and

$$g_0 = \frac{\Xi_2}{8z^7(-1+z)(z+1)(-1+\alpha)^4(-1+z^2-3z^2\alpha+3z^2\alpha^2)\alpha^3}$$

with $\Xi_2 = (-3z^4\alpha^2 + \alpha^2 W z^2 + 3z^2\alpha^2 + 3z^4\alpha - aWz^2 - 3z^2\alpha - z^4 + Wz^2 + 2z^2 - 1 - W)(-z^4\alpha^2 + 2z^4\alpha^3 + 1 - 3z^2\alpha^2 - W + \alpha^2 W z^2 - z^2 + 2z^2\alpha)(2z^2\alpha^2 - 2z^2\alpha + z^2 - 1 + W)$.

Plugging these results in and simplifying, we find explicit expressions for all four generating functions of interest, again we a common denominator $M$:

$M = (-1+z)(z+1)(-1+\alpha)^3(-1+z^2-3z^2\alpha+3z^2\alpha^2)(u^2 - z^2u^4\alpha - z^2u^2 + 2u^2\alpha z^2 + \alpha^2 z^2 u^4 - 2u^2\alpha^2 z^2 - z^2\alpha + z^2\alpha^2)$.

The first function:

$4z^6\alpha^3 M F_e = -(3z^4\alpha^2 - \alpha^2 W z^2 - 3z^2\alpha^2 - 3z^4\alpha + aWz^2 + 3z^2\alpha + z^4 - Wz^2 - 2z^2 + W + 1)(-z^4\alpha^4 - z^4\alpha^2 + 2z^4\alpha^3 - 23z^4u^2\alpha^2 + 12z^4u^2\alpha + 22z^4u^2\alpha^3 - 14\alpha^4 z^4 u^2 - 9z^4u^4\alpha^3 - 2z^4u^4\alpha + 6z^4u^4\alpha^2 + 5z^4u^4\alpha^4 - 3z^4u^2 - 6z^6\alpha^5 - 4z^6\alpha^3 + 7z^6\alpha^4 + 2\alpha^6 z^6 + z^6\alpha^2 - 6u^2z^6\alpha^5 - 22u^2z^6\alpha^3 + 16u^2z^6\alpha^4 + 5u^2\alpha^6 z^6 + 16u^2z^6\alpha^2 - 6u^2z^6\alpha - 5\alpha^6 z^6 u^4 - 13\alpha^4 z^6 u^4 + 11\alpha^5 z^6 u^4 + 11\alpha^3 z^6 u^4 - 5\alpha^2 z^6 u^4 + 2\alpha^7 z^8 u^4 + \alpha^5 z^8 u^4 - 3\alpha^6 z^8 u^4 + z^6 u^4\alpha + u^2 z^6 - 2u^2\alpha^7 z^8 + u^2\alpha^6 z^8 + z^2u^4\alpha + 3z^2u^2 - \alpha^2 z^2 u^4 - 6u^2\alpha z^2 + 7u^2\alpha^2 z^2 - u^2 + 4z^2u^2aW - z^2u^4aW + z^2u^4\alpha^2 W - 5u^2\alpha^2 z^2 W - 8u^2\alpha^3 W z^4 + 6u^2\alpha^4 W z^4 + 8u^2\alpha^2 W z^4 - 4u^2 W z^4\alpha - 3\alpha^2 z^4 u^4 W + 5\alpha^3 z^4 u^4 W - 3\alpha^4 z^4 u^4 W + \alpha^6 z^6 u^4 W + z^4u^4aW - z^6u^4\alpha^5 W - u^2\alpha^6 z^6 W + u^2 z^4 W + \alpha^2 W z^4 - 2\alpha^3 W z^4 - 2z^2u^2W + \alpha^4 W z^4 + u^2 W)$.

The second function:

$4z^5\alpha^3/(\alpha-1)/uMF_o = (-3z^4\alpha^2 + \alpha^2 W z^2 + 3z^2\alpha^2 + 3z^4\alpha - aWz^2 - 3z^2\alpha - z^4 + Wz^2 + 2z^2 - 1 - W)(1 + 12z^4\alpha^4 + 27z^4\alpha^2 - 25z^4\alpha^3 - 14z^4\alpha - z^6 - 7z^4u^2\alpha^2 + 2z^4u^2\alpha + 10z^4u^2\alpha^3 - 5\alpha^4 z^4 u^2 + 3z^4 + 8z^6\alpha^5 + 29z^6\alpha^3 - 22z^6\alpha^4 - 2\alpha^6 z^6 - 20z^6\alpha^2 + 7z^6\alpha + 5Wz^4\alpha - z^4 W - 5aWz^2 + 5\alpha^2 W z^2 + 2Wz^2 - W - 8u^2z^6\alpha^5 - 14u^2z^6\alpha^3 + 15u^2z^6\alpha^4 + 2u^2\alpha^6 z^6 + 6u^2z^6\alpha^2 - u^2z^6\alpha + 7z^2\alpha - 7z^2\alpha^2 - u^2\alpha z^2 + u^2\alpha^2 z^2 + z^2u^2aW - u^2\alpha^2 z^2 W - 6u^2\alpha^3 W z^4 + 3u^2\alpha^4 W z^4 + 4u^2\alpha^2 W z^4 - u^2 W z^4\alpha - 10\alpha^2 W z^4 + 9\alpha^3 W z^4 - 4\alpha^4 W z^4 - 3z^2)$.

The third function:

$8z^5\alpha^2 M G_e = (-3z^4\alpha^2 + \alpha^2 W z^2 + 3z^2\alpha^2 + 3z^4\alpha - aWz^2 - 3z^2\alpha - z^4 + Wz^2 + 2z^2 - 1 - W)(2\alpha^4 z^4 u^2 - 2\alpha^4 u^4 z^4 - 6z^4u^2\alpha^3 + 2z^4\alpha^3 + 4\alpha^3 u^4 z^4 - z^4\alpha^2 - 2\alpha^2 u^4 z^4 + \alpha^2 W z^2 + 6z^4u^2\alpha^2 - 3z^2\alpha^2 - 2z^4u^2\alpha + 2z^2\alpha + 1 - W - z^2)(2z^2\alpha^2 - 2z^2\alpha + z^2 - 1 + W)$.

The fourth function:

$8z^6(1-\alpha)\alpha^2/uMG_0 = (2z^2\alpha^2-2z^2\alpha+z^2-1+W)(2z^4u^2\alpha^3-2z^4\alpha^3-3z^4u^2\alpha^2+z^4\alpha^2-\alpha^2Wz^2+3z^2\alpha^2-u^2\alpha^2z^2+u^2\alpha^2z^2W+z^4u^2\alpha+u^2\alpha z^2-z^2u^2\alpha W-2z^2\alpha+z^2+W-1)(z\alpha+1)(-1+z\alpha)(-3z^4\alpha^2+\alpha^2Wz^2+3z^2\alpha^2+3z^4\alpha-\alpha Wz^2-3z^2\alpha-z^4+Wz^2+2z^2-1-W)$.

Of course, the expressions do not look appealing, but that is what they are. We can derive as many corollaries from this as we want, of course with Maple:

$$\begin{aligned}
f_0 &= [u^0]F_e = 1 + (2\alpha^2 + 1 - 2\alpha)z^2 + (5\alpha^4 - 10\alpha^3 + 9\alpha^2 - 4\alpha + 1)z^4 + \cdots ,\\
f_1 &= [u^1]F_o = \alpha z + (3\alpha^2 - 4\alpha + 2)\alpha z^3 + (8\alpha^4 - 19\alpha^3 + 20\alpha^2 - 11\alpha + 3)\alpha z^5 + \cdots ,\\
f_2 &= [u^2]F_e = \alpha(1-\alpha)z^2 + 2(1-\alpha)(2\alpha^2 + 1 - 2\alpha)\alpha z^4 + \cdots ,\\
f_3 &= [u^3]F_o = (1-\alpha)\alpha^2 z^3 + (1-\alpha)(5\alpha^2 - 6\alpha + 3)\alpha^2 z^5 + \cdots .
\end{aligned}$$

and similarly

$$\begin{aligned}
g_0 &= [u^0]G_e = \alpha(1-\alpha)^2 z^3 + (5\alpha^2 - 4\alpha + 2)(1-\alpha)^2\alpha z^5 + \cdots ,\\
g_1 &= [u^1]G_o = \alpha(1-\alpha)z^2 + 2(1-\alpha)(2\alpha^2 + 1 - 2\alpha)\alpha z^4,\\
g_2 &= [u^2]G_e = (1-\alpha)\alpha^2 z^3 + (1-\alpha)(5\alpha^2 - 6\alpha + 3)\alpha^2 z^5 + \cdots ,\\
g_3 &= [u^3]G_o = \alpha^2(1-\alpha)^2 z^4 + 3\alpha^2(2\alpha^2 + 1 - 2\alpha)(1-\alpha)^2 z^6 + \cdots .
\end{aligned}$$

## 3. A more sophisticated approach

The imbalance of $\alpha$ versus $\beta$ is leveled out after 2 (or an even number of) steps. Thus, as in [3], we consider the system after an even number of steps. In the following graph, a directed arrow stands for 2 steps (a double-step). Note that the system is still working without look-ahead, writing s for the small item of size $\frac{1}{3}$ and l for the large item of size $\frac{2}{3}$, the sequences sl resp. ls lead to different states when being in the special state named $Q$.

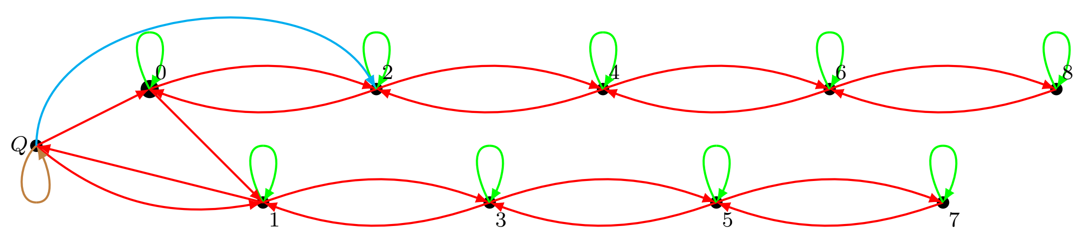


FIGURE 4. Two steps. Red with probability $\alpha\beta$, green with probability $1-2\alpha\beta = \alpha^2+\beta^2$, blue with probability $\beta^2$, brown with probability $\alpha^2$.

The graph is now simpler than before. We introduce generating functions $f_N$ for the upper layer of states, and $g_N$ for the lower layer of states. The meaning of these generating functions is now different from the previous section, but it is apparent how they are related. Here are the recursions:

$$\begin{aligned}
f_N &= z\alpha\beta f_{N-1} + z\alpha\beta f_{N+1} + z(\alpha^2+\beta^2)f_N, \quad N \ge 2,\\
f_1 &= z\alpha\beta f_0 + z\alpha\beta f_2 + z(\alpha^2+\beta^2)f_1 + z\beta^2 f_Q,\\
f_0 &= 1 + z\alpha\beta f_1 + z(\alpha^2+\beta^2)f_0 + z\alpha\beta f_Q,
\end{aligned}$$

$$\begin{aligned}
f_Q &= z\alpha\beta g_0 + z\alpha^2 f_Q = \frac{z\alpha\beta g_0}{1-z\alpha^2},\\
g_N &= z\alpha\beta g_{N-1} + z\alpha\beta g_{N+1} + z(\alpha^2+\beta^2)g_N, \quad N\ge 1,\\
g_0 &= z\alpha\beta f_0 + z\alpha\beta g_1 + z\alpha\beta f_Q + z(\alpha^2+\beta^2)g_0.
\end{aligned}$$

Introducing only *two* bivariate generating functions

$$F(u) = \sum_{N\ge 0} u^N f_N \quad\text{and}\quad G(u) = \sum_{N\ge 0} u^N g_N,$$

we find by summing the recursions

$$\begin{aligned}
F(u) = \sum_{N\ge 0} u^N f_N &= z\alpha\beta \sum_{N\ge 2} u^N f_{N-1} + z\alpha\beta \sum_{N\ge 2} u^N f_{N+1} + z(\alpha^2+\beta^2)\sum_{N\ge 2} u^N f_N\\
&+ u(z\alpha\beta f_0 + z\alpha\beta f_2 + z(\alpha^2+\beta^2)f_1 + z\beta^2\frac{z\alpha\beta g_0}{1-z\alpha^2})\\
&+ 1 + z\alpha\beta f_1 + z(\alpha^2+\beta^2)f_0 + z\alpha\beta\frac{z\alpha\beta g_0}{1-z\alpha^2}\\
&= z\alpha\beta u F(u) + \frac{z\alpha\beta}{u}(F(u)-f_0) + z(\alpha^2+\beta^2)F(u)\\
&+ uz^2\beta^3\alpha\frac{g_0}{1-z\alpha^2} + 1 + z^2\alpha^2\beta^2\frac{g_0}{1-z\alpha^2}
\end{aligned}$$

and

$$\begin{aligned}
G(u) = \sum_{N\ge 0} u^N g_N &= z\alpha\beta \sum_{N\ge 1} u^N g_{N-1} + z\alpha\beta \sum_{N\ge 1} u^N g_{N+1} + z(\alpha^2+\beta^2)\sum_{N\ge 1} u^N g_N\\
&+ z\alpha\beta f_0 + z\alpha\beta g_1 + z\alpha\beta\frac{z\alpha\beta g_0}{1-z\alpha^2} + z(\alpha^2+\beta^2)g_0\\
&= z\alpha\beta u G(u) + \frac{z\alpha\beta}{u}(G(u)-g_0) + z(\alpha^2+\beta^2)G(u)\\
&+ z\alpha\beta f_0 + z^2\alpha^2\beta^2\frac{g_0}{1-z\alpha^2}.
\end{aligned}$$

Solving the system leads to

$$\begin{aligned}
F(u) &= \frac{-uz\alpha^2 + z^2\alpha^2\beta^2 g_0 u - z\alpha\beta f_0 + z^2\alpha^3\beta f_0 + u + u^2 z^2\beta^3\alpha g_0}{u - 2uz\alpha^2 - z\alpha\beta u^2 + z^2\alpha^3\beta u^2 - z\alpha\beta + z^2\alpha^3\beta + z^2 u\alpha^4 - zu\beta^2 + z^2 u\beta^2\alpha^2},\\
G(u) &= -\frac{z\alpha\beta(-z\alpha\beta g_0 u + g_0 - g_0 z\alpha^2 + z\alpha^2 f_0 u - f_0 u)}{u - 2uz\alpha^2 - z\alpha\beta u^2 + z^2\alpha^3\beta u^2 - z\alpha\beta + z^2\alpha^3\beta + z^2 u\alpha^4 - zu\beta^2 + z^2 u\beta^2\alpha^2}.
\end{aligned}$$

These answers are implicit, since they contain $f_0 = F(0)$ and $g_0 = G(0)$. To make them explicit, the kernel method is used once again. The denominators factor as

$$z\alpha\beta(-1+z\alpha^2)(u-r_1)(u-r_2)$$

with

$$r_2 = \frac{1 - z\alpha^2 - z\beta^2 - \sqrt{z^2\alpha^4 - 2z^2\beta^2\alpha^2 - 2z\alpha^2 + z^2\beta^4 - 2z\beta^2 + 1}}{2z\alpha\beta}$$

and $r_1 = \frac{1}{r_2}$.

The factor $(u-r_2)$ (the 'bad' factor) must cancel from numerator and denominator. The result is now

$$F(u)=\frac{r_2z^2\beta^3\alpha g_0-z\alpha^2+z^2\alpha^2\beta^2g_0+1+uz^2\beta^3\alpha g_0}{z\alpha\beta(-1+z\alpha^2)(u-r_1)}$$

and

$$G(u)=\frac{-(-z\alpha\beta g_0+z\alpha^2f_0-f_0)}{(-1+z\alpha^2)(u-r_1)},$$

Plugging in $u=0$, we get

$$f_0=\frac{r_2z^2\beta^3\alpha g_0-z\alpha^2+z^2\alpha^2\beta^2g_0+1}{z\alpha\beta(-1+z\alpha^2)(-r_1)},$$

$$g_0=\frac{(-z\alpha\beta g_0+z\alpha^2f_0-f_0)}{(-1+z\alpha^2)r_1}.$$

From these, we can compute $f_0$ and $g_0$ easily, but don't print it, since it is not too attractive at the moment (in a moment, it will become very beautiful).

It is easy to see that

$$[u^j]G(u)=\frac{(-z\alpha\beta g_0+z\alpha^2f_0-f_0)}{(-1+z\alpha^2)}r_2^{j+1}$$

and

$$[u^j]F(u)=-r_2^{j+1}\frac{r_2z^2\beta^3\alpha g_0-z\alpha^2+z^2\alpha^2\beta^2g_0+1}{z\alpha\beta(-1+z\alpha^2)}-r_2^j\frac{z\beta^2g_0}{(-1+z\alpha^2)}.$$

Note that $[z^mu^j]F(u)$ is the probability to reach state $2j$ in $m$ (double-)steps, and $[z^mu^j]G(u)$ is the probability to reach state $2j+1$ in $m$ (double-)steps.

**More attractive formulæ thanks to a substitution.** Using the substitution

$$z=\frac{v}{\alpha\beta+(\alpha^2+\beta^2)v+\alpha\beta v^2}=\frac{v}{(\alpha+v\beta)(\beta+v\alpha)},$$

(inspired by our old paper [3]) all the expressions become nicer. For instance, $r_2=v$ and

$$f_0=\frac{(v\alpha+\beta)(\alpha+v\beta)}{\alpha\beta(1-v)(v^2+v+1)},$$

$$g_0=\frac{v(\alpha+\alpha v^2+v\beta)(v\alpha+\beta)}{\alpha\beta(1-v)(v^2+v+1)}.$$

The equality $(1-v)(v^2+v+1)=1-v^3$ might be useful as well. Even the full bivariate generating functions look now very nice:

$$F=\frac{(uv^3\beta+\alpha+v\beta)(v\alpha+\beta)}{\beta\alpha(1-uv)(1-v)(v^2+v+1)}$$

$$G=\frac{v(\alpha+\alpha v^2+v\beta)(v\alpha+\beta)}{\beta\alpha(1-uv)(1-v)(v^2+v+1)}.$$

Consequently, reading off coefficient of powers of $u$,

$$[u^j]F=\frac{v^j(\alpha+v\beta)(v\alpha+\beta)}{\beta\alpha(1-v)(v^2+v+1)}+\frac{v^{j+1}}{\alpha(1-v)(v^2+v+1)}$$

and

$$[u^j]G = \frac{v^{j+1}(\alpha + \alpha v^2 + v\beta)(v\alpha + \beta)}{\beta\alpha(1-v)(v^2+v+1)}.$$

Finally we answer the question how to read off coefficients of powers of $z$ when the function is given in terms of $v$: For that, we employ Cauchy's integral formula:

$$\begin{aligned}
[z^N]H(z(v)) &= \frac{1}{2\pi i}\oint \frac{dz}{z^{N+1}} H(z(v)) \\
&= \frac{1}{2\pi i}\oint \frac{dv}{v^{N+1}} \frac{\alpha\beta(1-v^2)}{(\alpha+\beta v)(\beta+\alpha v)} (\alpha+\beta v)^{N+1}(\beta+\alpha v)^{N+1} H(v) \\
&= [v^N]\alpha\beta(1-v^2)(\alpha+\beta v)^N(\beta+\alpha v)^N H(v).
\end{aligned}$$

**Odd number of steps.** For that, we don't need to do new calculations, by considering the last step separately. We refer to the original Figure 3. It is immediate to see that

$$\begin{aligned}
&\mathbb{P}\{\text{reach top level state } 2j+1 \text{ in } 2m+1 \text{ steps}\} \\
&\qquad = \alpha\mathbb{P}\{\text{reach top level state } 2j \text{ in } 2m \text{ steps}\} \\
&\qquad + \beta\mathbb{P}\{\text{reach top level state } 2j+2 \text{ in } 2m \text{ steps}\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\{\text{reach bottom level state } 2j \text{ in } 2m+1 \text{ steps}\} \\
&\qquad = \alpha\mathbb{P}\{\text{reach bottom level state } 2j-1 \text{ in } 2m \text{ steps}\} \\
&\qquad + \beta\mathbb{P}\{\text{reach bottom level state } 2j+1 \text{ in } 2m \text{ steps}\};
\end{aligned}$$

the exceptional cases near the beginning are easy to figure out directly.

## 4. Conclusion

We want to emphasize the following points:

- A brute-force approach is possible, but leads to equations of order 4 and explicit but very ungainly expressions.
- Looking at the system after an even number of steps is a clever idea, since the imbalance of $\alpha$ versus $\beta$ is leveled out. The equations are only quadratic.
- Introducing an auxiliary variable, all the generating functions become rational (in the variable). Consequently reading off coefficients is not difficult.
- To go from an even number of steps to an odd number of steps is not difficult, when considering the last step separately and use previous results.

## References

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