MIN-TURNS AND MAX-TURNS IN k-DYCK PATHS: A PURE GENERATING FUNCTION APPROACH

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A christmas present for all k-Dyck paths lovers.

Abstract. k-Dyck paths differ from ordinary Dyck paths by using an up-step of length k. We analyze at which level the path is after the s-th up-step and before the $(s+1)$st up-step. In honour of Rainer Kemp who studied a related concept 40 years ago the terms MAX-terms and MIN-terms are used. Results are obtained by an appropriate use of trivariate generating functions; practically no combinatorial arguments are used.

1. Introduction

Our objects are k-Dyck paths, having up-steps $(1, k)$ and down-steps $(1, -1)$, and never go below the x-axis. At the end, they reach the x-axis, but we also need versions that end at a prescribed level different from 0. Much material about such paths can be found in [6].

We consider MAX-turns, where each up-step ends and MIN-turns, where each up-step starts. The figure 1 explains the concept readily:

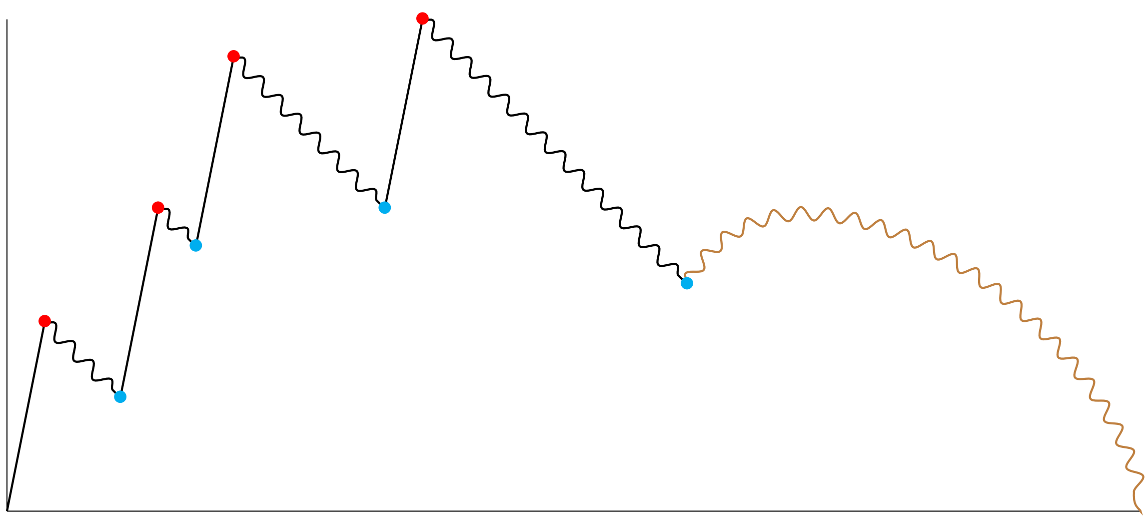

Figure 1. The first 4 MAX-turns and the first 4 MIN-turns are shown.

k-Dyck paths can only exist for a length of the form $(k + 1)N$, which is clear for combinatorial reasons or otherwise. We want to know the average level of the s-th MAX-turn resp. MIN-turn, amoung all k-Dyck of the same length. In order to do this,

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we sum the level of the s-th MAX-turn resp. MIN-turn, over all k-Dyck paths of length $(k+1)N$. To get the average, one only needs to divide by the total number of such k-Dyck paths.

Our main achievment is to get fully explicit functions

$$\textsc{max}(z,w) = \sum_{N\ge 0,\, s\ge 1} z^{(k+1)N} w^s [\text{cumulative level of the } s\text{-th MAX-turn in all } k\text{-Dyck paths of length } (k+1)N]$$

and

$$\textsc{min}(z,w) = \sum_{N\ge 0,\, s\ge 1} z^{(k+1)N} w^s [\text{cumulative level of the } s\text{-th MIN-turn in all } k\text{-Dyck paths of length } (k+1)N].$$

As a bonus we get $\textsc{osc}(z,w) := \textsc{max}(z,w) - \textsc{min}(z,w)$; this cumulates the lengths of the wavy line between the s-th MAX-turn and s-th MIN-turn, which is a somewhat simpler function. In this way, we recover some of the results from [1] without resorting to any bijective combinatorics. Note that such a wavy line might have length zero as well, if two up-steps follow each other immediately. Rainer Kemp in [4] has considered MAX- and MIN-turns for Dyck paths, although his definitions were slightly different (peaks and valleys).

The key to the success of our method is the simple but perhaps unusual identity

$$\sum_{i\ge 0} \big([z^i]f(z)\big)\cdot y^i = f(y).$$

2. Some basic observations

As we will see soon, the equation $u = z + zwu^{k+1}$ plays a major role when enumerating k-Dyck paths. The equation is of the form $u = z\Phi(u)$, with $\Phi(u) = 1 + wu^{k+1}$, so it is amenable to the Lagrange inversion [3], and

$$\begin{aligned}[z^{(k+1)N+1}]u &= \frac{1}{(k+1)N+1}[u^{(k+1)N}](1+wu^{k+1})^{(k+1)N+1}\\ &= \frac{1}{(k+1)N+1} w^N \binom{(k+1)N+1}{N}.\end{aligned}$$

We will need the formula

$$\overline{u} = z\sum_{N\ge 0} w^N z^{(k+1)N} \frac{1}{kN+1}\binom{(k+1)N}{N}. \tag{1}$$

We also need the version where $w = 1$, so $\widehat{u}$ satisfies the equation $u = z + zu^{k+1}$:

$$\widehat{u} = z\sum_{\lambda\ge 0} \frac{1}{(k+1)\lambda+1}\binom{1+(k+1)\lambda}{\lambda} z^{(k+1)\lambda} \tag{2}$$

and

$$\widehat{u}^{-k} = z^{-k} - z\sum_{\lambda\ge 0}\frac{k}{\lambda+1}\binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda}. \tag{3}$$

3. MIN-TURNS

The following substitution is essential for adding a new slice (an up-step, followed by a maximal sequence of down-steps):

$$u^j \longrightarrow zw \sum_{0\le i\le j+k} z^i u^{j+k-i} = \frac{zwu^{k+1}}{u-z}u^j - \frac{wz^{k+2}}{u-z}z^j$$

The technique of adding-a-new slice is described in [2].

Now let $F_m(u) = F_m(u;z)$ be the generating function according to m slices; z refers to the lengths and u to the level of the m-th MIN-turn. The substitution leads to

$$F_{m+1}(u) = \frac{zwu^{k+1}}{u-z}F_m(u) - \frac{wz^{k+2}}{u-z}F_m(z), \quad F_0(u) = 1.$$

Let $F(u) = \sum_{m\ge0} F_m(u)$, so that we don't care about the number m anymore, since the variable w takes care of it; then

$$F(u) = F(u;z,w) = 1 + \frac{zwu^{k+1}}{u-z}F(u) - \frac{wz^{k+2}}{u-z}F(z),$$

or

$$F(u) = \frac{u-z-wz^{k+2}F(z)}{u-z-zwu^{k+1}}.$$

$u = \overline{u}$ is a factor of the denominator, but for z and u small, we have $u \sim z$, so this factor must cancel in the numerator as well. This is what one learns from the kernel method [5].

We find

$$F(z) = \frac{\overline{u}-z}{wz^{k+2}}$$

and further

$$F(u) = F(u;z,w) = \frac{u-\overline{u}}{u-z-zwu^{k+1}}.$$

That ends the computation of the "left" part of the k-Dyck path. For the right one, we start at level h with an up-step and end eventually at the zero level. We don't use the variable w here. The kernel method could also be used, but there is a simpler way. Reading the path from right to left, there is a decomposition when the path leaves a level and never comes back to it. Note that $\overline{u}/z$ is the generating function of k-Dyck paths. With this, we find

$$\Big(\frac{\overline{u}}{z}\Big)^h z^h \Big(\frac{\overline{u}}{z} - 1\Big) = \frac{(\overline{u}-z)\overline{u}^h}{z};$$

the minus 1 term happens since the reversed path must end with a down-step. The formula works only for $h \ge 1$, but the instance $h = 0$ is not needed (although easy). The fact that this generating function is essentially a power is part of our successful approach.

We now have

$$\begin{aligned}\text{MIN}(z,w) &= \sum_{h\ge1} h[u^h]F(u)\cdot\frac{(\overline{u}-z)\overline{u}^h}{z}\\ &= \sum_{h\ge1}[u^{h-1}]\frac{d}{du}F(u)\cdot\frac{(\overline{u}-z)\overline{u}^h}{z}\end{aligned}$$

$$= \frac{(\overline{u}-z)\overline{u}}{z}\cdot\frac{-zu+uzwu^{k+1}k+\overline{u}u-\overline{u}zwu^{k+1}k-\overline{u}zwu^{k+1}}{u(u-z-zwu^{k+1})^2}\bigg|_{u=\widehat{u}}$$
$$= \frac{(\overline{u}-z)\overline{u}}{z}\cdot\frac{-z\widehat{u}+uzw\widehat{u}^{k+1}k+\overline{u}\widehat{u}-\overline{u}zw\widehat{u}^{k+1}k-\overline{u}zw\widehat{u}^{k+1}}{\widehat{u}(\widehat{u}-z-zw\widehat{u}^{k+1})^2}.$$

A simplification that only uses $\widehat{u}^{k+1}=\frac{\widehat{u}-z}{z}$ eventually leads to

$$\textsc{min}(z,w)=\frac{kw\widehat{u}}{z(1-w)^2}+\frac{(\overline{u}-z)}{z^2(1-w)^2\widehat{u}^k}-\frac{(k+1)\overline{u}w}{z(1-w)^2}.$$

Theorem 1. *The generating funtion* $\textsc{min}(z,w)$ *where the coefficient of* $z^{(k+1)N}u^s$ *refers to the cumulative levels of the s-th* $\textsc{min}$*-turn, is given by*

$$\textsc{min}(z,w)=\frac{kw\widehat{u}}{z(1-w)^2}+\frac{(\overline{u}-z)}{z^2(1-w)^2\widehat{u}^k}-\frac{(k+1)\overline{u}w}{z(1-w)^2}.$$

The next step is to expand this function:

$$\begin{aligned}
[w^s]\textsc{min}(z,w) &= [w^s]\frac{kw\widehat{u}}{z(1-w)^2}-[w^s]\frac{(k+1)\overline{u}w}{z(1-w)^2}+[w^s]\frac{(\overline{u}-z)}{z^2(1-w)^2\widehat{u}^k}\\
&= \frac{sk\widehat{u}}{z}-(k+1)\sum_{i=0}^{s-1}(s-i)[w^i]\frac{\overline{u}}{z}\\
&+\sum_{i=0}^{s}(s+1-i)[w^i]\frac{(\overline{u}-z)}{z^2\widehat{u}^k}\\
&= \frac{sk\widehat{u}}{z}-(k+1)\sum_{i=0}^{s-1}(s-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\\
&+\sum_{i=1}^{s}(s+1-i)\frac{1}{z\widehat{u}^k}z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\\
&= \frac{sk\widehat{u}}{z}-(k+1)\sum_{i=0}^{s-1}(s-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\\
&+\sum_{i=1}^{s}(s+1-i)z^{(k+1)(i-1)}\frac{1}{ki+1}\binom{(k+1)i}{i}\\
&-\sum_{i=1}^{s}(s+1-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\sum_{\lambda\ge0}\frac{k}{\lambda+1}\binom{(k+1)\lambda}{\lambda}z^{(k+1)\lambda}.
\end{aligned}$$

And now we read off the coefficient of $z^{(k+1)N}$; we assume that $N\ge s$, otherwise a path would not have an s-th $\textsc{min}$-turn:

$$\begin{aligned}
[w^sz^{(k+1)N}]\textsc{min}(z,w) &= sk\frac{1}{kN+1}\binom{(k+1)N}{N}\\
&-\sum_{i=1}^{s}(s+1-i)\frac{1}{ki+1}\binom{(k+1)i}{i}\frac{k}{(N-i)+1}\binom{(k+1)(N-i)}{(N-i)}
\end{aligned}$$

Theorem 2. *The sum of levels of the s-th* MIN*-turns in all the k-Dyck paths of length $(k+1)N$ is given by*

$$\frac{sk}{kN+1}\binom{(k+1)N}{N}-\sum_{i=1}^{s}(s+1-i)\frac{1}{ki+1}\binom{(k+1)i}{i}\frac{k}{(N-i)+1}\binom{(k+1)(N-i)}{(N-i)}.$$

4. MAX-TURNS

It is easy to go from a MIN-turn to the next MAX-turn, just by doing one up-step. On the level of generating functions, this means

$$G(u;z,w)=F(u;z,w)wzu^k.$$

The right side is even easier than before, since level h must be reached without further restriction. Result:

$$\Big(\frac{\widehat{u}}{z}\Big)^{h+1}z^h=\widehat{u}^{h-1}\frac{\widehat{u}^2}{z}.$$

$$\begin{aligned}\mathrm{MAX}(z,w)&=\sum_{h\ge1}h[u^h]G(u)\cdot\widehat{u}^{h-1}\frac{\widehat{u}^2}{z}=\frac{\widehat{u}^2}{z}\sum_{h\ge1}[u^{h-1}]\frac{d}{du}G(u)\cdot\widehat{u}^{h-1}\\&=\frac{wk\widehat{u}}{z(1-w)^2}-\frac{wk\overline{u}}{z(1-w)^2}+\frac{w(\overline{u}-z)}{z^2\widehat{u}^k(1-w)^2}-\frac{w^2\overline{u}}{z(1-w)^2}.\end{aligned}$$

The same type of simplifications as before have been applied.

And now we go to the coefficients of this:

$$\begin{aligned}[w^s]\mathrm{MAX}(z,w)&=s\frac{k\widehat{u}}{z}-\sum_{i=0}^{s-1}(s-i)[w^i]\frac{k\overline{u}}{z}\\&\quad+\frac{1}{z\widehat{u}^k}\sum_{i=0}^{s-1}(s-i)[w^i]\frac{(\overline{u}-z)}{z}-\sum_{i=0}^{s-2}(s-1-i)[w^i]\frac{\overline{u}}{z}\\&=sk\sum_{N\ge0}z^{(k+1)N}\frac{1}{kN+1}\binom{(k+1)N}{N}\\&\quad-\sum_{i=0}^{s-1}(s-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\\&\quad+\sum_{i=1}^{s-1}(s-i)z^{(k+1)(i-1)}\frac{1}{ki+1}\binom{(k+1)i}{i}\\&\quad-\sum_{\lambda\ge0}\frac{k}{\lambda+1}\binom{(k+1)\lambda}{\lambda}z^{(k+1)\lambda}\sum_{i=1}^{s-1}(s-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}\\&\quad-\sum_{i=0}^{s-2}(s-1-i)z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}.\end{aligned}$$

And, again for $N \ge s$, we read off the coefficient of $z^{(k+1)N}$:

$$\begin{aligned}[w^s z^{(k+1)N}]\mathrm{MAX}(z,w) &= sk\frac{1}{kN+1}\binom{(k+1)N}{N}\\ &\qquad -\sum_{i=1}^{s-1}(s-i)\frac{1}{ki+1}\binom{(k+1)i}{i}\frac{k}{N-i+1}\binom{(k+1)(N-i)}{N-i}.\end{aligned}$$

Theorem 3. *The generating function* $\mathrm{MAX}(z,w)$ *is given by*

$$\mathrm{MAX}(z,w) = \frac{wk\widehat{u}}{z(1-w)^2} - \frac{wk\overline{u}}{z(1-w)^2} + \frac{w(\overline{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{w^2\overline{u}}{z(1-w)^2}.$$

The coefficient $[w^s z^{(k+1)N}]\mathrm{MAX}(z,w)$ *is for* $N \ge s$ *given by*

$$\frac{sk}{kN+1}\binom{(k+1)N}{N} - \sum_{i=1}^{s-1}(s-i)\frac{1}{ki+1}\binom{(k+1)i}{i}\frac{k}{N-i+1}\binom{(k+1)(N-i)}{N-i}.$$

5. The oscillation

The cumulative function of the s-th oscillation (total length of the s-th wavy line in all paths of length $(k+1)N$) is

$$\mathrm{OSC}(z,w) := \mathrm{MAX}(z,w) - \mathrm{MIN}(z,w).$$

There are some cancellations and simplifications that we don't show here, as there are no special skills needed to get the result

$$\mathrm{OSC}(z,w) = \frac{\overline{u}w}{z(1-w)} - \frac{(\overline{u}-z)}{z^2\widehat{u}^k(1-w)}.$$

Further,

$$\begin{aligned}[w^s]\mathrm{OSC}(z,w) &= \sum_{i=1}^{s}[w^i]\frac{\overline{u}}{z} - \sum_{i=0}^{s}[w^i]\frac{(\overline{u}-z)}{z^2\widehat{u}^k}\\ &= \sum_{i=1}^{s} z^{(k+1)i}\frac{1}{kN+1}\binom{(k+1)i}{i} - \frac{1}{z\widehat{u}^k}\sum_{i=0}^{s}[w^i]\frac{(\overline{u}-z)}{z}\\ &= \sum_{i=1}^{s} z^{(k+1)i}\frac{1}{kN+1}\binom{(k+1)i}{i}\\ &\quad - \sum_{i=1}^{s} z^{(k+1)(i-1)}\frac{1}{ki+1}\binom{(k+1)i}{i}\\ &\quad + \sum_{\lambda\ge 0}\frac{k}{\lambda+1}\binom{(k+1)\lambda}{\lambda}z^{(k+1)\lambda}\sum_{i=1}^{s} z^{(k+1)i}\frac{1}{ki+1}\binom{(k+1)i}{i}.\end{aligned}$$

Finally, for $N \ge s$, we look at $[w^s z^{(k+1)N}]\mathrm{OSC}(z,w)$ and obtain

$$\sum_{i=1}^{s}\frac{1}{ki+1}\binom{(k+1)i}{i}\frac{k}{N-i+1}\binom{(k+1)(N-i)}{N-i}.$$

This has been obtained in [1] by other methods.

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