LATTICE PATHS WITH INFINITELY MANY DOWN STEPS – THE NEGATIVE BOUNDARY MODEL

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Abstract. We consider a variation of Dyck paths, where additionally to steps $(1,1)$ and $(1,-1)$ down-steps $(1,-j)$, for $j \ge 2$ are allowed. We give credits to Emeric Deutsch for that. The enumeration of such objects living in a strip is performed. Methods are the kernel method and techniques from linear algebra.

1. Introduction

Emeric Deutsch [1] had the idea to consider a variation of ordinary Dyck paths, by augmenting the usual up-steps and down-steps by one unit each, by down-steps of size $3, 5, 7, \ldots$. This leads to ternary equations, as can be seen for instance from [3].

The present author started to investigate a related but simpler model of down-steps $1, 2, 3, 4, \ldots$ and investigated it (named Deutsch paths in honour of Emeric Deutsch) in a series of papers, [2, 4, 5].

This paper is an further member of this series: The condition that (as with Dyck paths) the paths cannot enter negative territory, is relaxed, by introducing a negative boundary $-t$. Here are two recent publications about such a negative boundary: [8] and [7].

Instead of allowing negative altitudes, we think about the whole system shifted up by t units, and start at the point $(0,t)$ instead. This is much better for the generating functions that we are going to investigate. Eventually, the results can be re-interpreted as results about enumerations with respect to a negative boundary.

The setting with flexible initial level t and final level j allows us to consider the Deutsch paths also from left to right (they are not symmetric!), without any new computations.

The next sections achieves this, using the celebrated kernel-method, one of the tools that is dear to our heart [6].

In the following section, an additional upper bound is introduced, so that the Deutsch paths live now in a strip. The way to attack this is linear algebra. Once everything has been computated, one can relax the conditions and let lower/upper boundary go to $\mp\infty$.

2. Generating functions and the kernel method

As discussed, we consider Deutsch paths starting at $(0,t)$ and ending at (n,j), for $n, t, j \ge 0$. First we consider univariate generating functions $f_j(z)$, where z^n stays for

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n steps done, and j is the final destination. The recursion is immediate:

$$f_j(z) = [\![t=j]\!] + zf_{j-1}(z) + z\sum_{k>j} f_k(z),$$

where $f_{-1}(z) = 0$. Next, we consider

$$F(z,u) := \sum_{j\ge 0} f_j(z)u^j,$$

and get

$$\begin{aligned} F(z,u) &= u^t + zuF(z,u) + z\sum_{j\ge 0} u^j \sum_{k>j} f_k(z) \\ &= u^t + zuF(z,u) + z\sum_{k>0} f_k(z) \sum_{0\le j<k} u^j \\ &= u^t + zuF(z,u) + z\sum_{k\ge 0} f_k(z)\frac{1-u^k}{1-u} \\ &= u^t + zuF(z,u) + \frac{z}{1-u}[F(z,1)-F(z,u)] \\ &= \frac{u^t(1-u)+zF(z,1)}{z-zu+zu^2+1-u}. \end{aligned}$$

Since the critical value is around $u=1$, we write the denominator as

$$z(u-1)^2 + (u-1)(z-1) + z = z(u-1-r_1)(u-1-r_2),$$

with

$$r_1 = \frac{1-z+\sqrt{1-2z-3z^2}}{2z}, \quad r_2 = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}.$$

The factor $(u-1-r_2)$ is bad, so the numerator must vanish for $[u^t(1-u)+zF(z,1)]\big|_{u=1+r_2}$, therefore

$$zF(z,1) = (1+r_2)^t r_2.$$

Furthermore

$$F(z,u) = \frac{\frac{u^t(1-u)+zF(z,1)}{u-r_2}}{z(u-r_1)}.$$

The expressions become prettier using the substitution $z = \frac{v}{1+v+v^2}$; then

$$r_1 = \frac{1}{v}, \quad r_2 = v.$$

It can be proved by induction (or computer algebra) that

$$\frac{u^t(1-u)+v(1+v)^t}{u-1-v} = -v\sum_{k=0}^{t-1}(1+v)^{t-1-k} - u^t.$$

Furthermore

$$\frac{1}{z(u-1-r_1)} = -\frac{1}{z(1+r_1)(1-\frac{u}{1+r_1})},$$

and so

$$f_j(z) = [u^j]F(z,u) = [u^j]\Big[v\sum_{k=0}^{t-1}(1+v)^{t-1-k}u^k + u^t\Big]\sum_{\ell\ge 0}\frac{u^\ell}{z(1+r_1)^{\ell+1}}.$$

Of interest are two special cases: The case that was studied before [2] is $t=0$:

$$f_j = \frac{(1+v+v^2)v^j}{(1+v)^{j+1}}.$$

The other special case is $j=0$ for general t, as it may be interpreted as Deutsch paths read from right to left, starting at level 0 and ending at level $t\ge 1$ (for $t=0$, the previous formula applies):

$$\begin{aligned}f_0(z) &= [u^0]\Big[v\sum_{k=0}^{t-1}(1+v)^{t-1-k}u^k + u^t\Big]\sum_{\ell\ge 0}\frac{u^\ell}{z(1+r_1)^{\ell+1}}\\ &= v(1+v)^{t-1}\frac{1}{z(1+r_1)} = v(1+v+v^2)(1+v)^{t-2}.\end{aligned}$$

The next section will present a simplification of the expression for $f_j(z)$, which could be obtained directly by distinguishing cases and summing some geometric series.

3. Refined analysis: lower and upper boundary

Now we consider Deutsch paths bounded from below by zero and bounded from above by $m-1$; they start at level t and end at level j after n steps. For that, we use generating functions $\varphi_j(z)$ (the quantity t is a silent parameter here). The recursions that are straight-forwarded are best organized in a matrix, as the following example shows.

$$\begin{pmatrix}1 & -z & -z & -z & -z & -z & -z & -z\\ -z & 1 & -z & -z & -z & -z & -z & -z\\ 0 & -z & 1 & -z & -z & -z & -z & -z\\ 0 & 0 & -z & 1 & -z & -z & -z & -z\\ 0 & 0 & 0 & -z & 1 & -z & -z & -z\\ 0 & 0 & 0 & 0 & -z & 1 & -z & -z\\ 0 & 0 & 0 & 0 & 0 & -z & 1 & -z\\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \varphi_5\\ \varphi_6\\ \varphi_7\end{pmatrix} = \begin{array}{l}\begin{pmatrix}0\\ 0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\end{array}\begin{array}{l}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}t\\ \\ \\ \\ \\ \\ \end{array}$$

The goal is now to solve this system. For that the substitution $z=\frac{v}{1+v+v^2}$ is used throughout. The method is to use Cramer's rule, which means that the right-hand side has to replace various columns of the matrix, and determinants have to be computed. At the end, one has to divide by the determinant of the system.

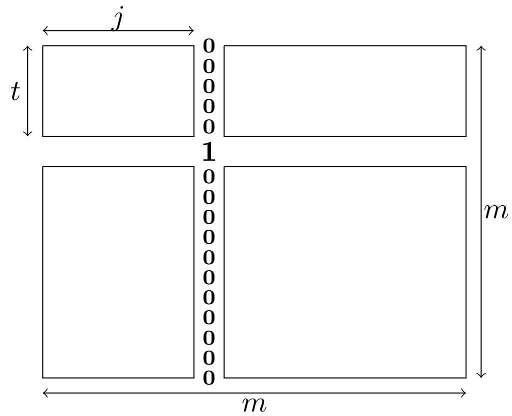
Let D_m be the determinant of the matrix with m rows and columns. The recursion

$$(1+v+v^2)^2 m_{n+2} - (1+v+v^2)(1+v)^2 D_{m+1} + v(1+v)^2 D_m = 0$$

appeared already in [2] and is not difficult to derive and to solve:

$$D_m = \frac{(1+v)^{m-1}}{(1+v+v^2)^m}\frac{1-v^{m+2}}{1-v}.$$

To solve the system with Cramer's rule, we must compute a determinant of the following type,



where the various rows are replaced by the right-hand side. While it is not impossible to solve this recursion by hand, it is very easy to make mistakes, so it is best to employ a computer. Let $D(m;t,j)$ the determinant according to the drawing.

It is not unexpected that the results are different for $j<t$ resp. $j\ge t$. Here is what we found:

$$D(m;t,j)=\frac{(1+v)^{t-j-3+m}(1-v^{j+1})v(1-v^{m-t})}{(1-v)^2(1+v+v^2)^{m-1}},\quad \text{for } j<t,$$

$$D(m;t,j)=\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})}{(1-v)^2(1+v+v^2)^{m-1}(1+v)^{j-t+3-m}},\quad \text{for } j\ge t.$$

To solve the system, we have to divide by the determinant D_m, with the result

$$\varphi_j=\frac{D(m;t,j)}{D_m}=\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1-v^{m-t})(1+v+v^2)}{(1-v)(1-v^{m+2})},\quad \text{for } j<t,$$

$$\varphi_j=\frac{D(m;t,j)}{D_m}=\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}(1-v^{m+2})},\quad \text{for } j\ge t.$$

We found all this using Computer algebra. Some critical minds may argue that this is only experimental. One way of rectifying this would be to show that indeed the functions φ_j solve the system, which consists of summing various geometric series; again, a computer could be helpful for such an enterprise.

Of interest are also the limits for $m\to\infty$, i.e., no upper boundary:

$$\varphi_j=\lim_{m\to\infty}\frac{D(m;t,j)}{D_m}=\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1+v+v^2)}{(1-v)},\quad \text{for } j<t,$$

$$\varphi_j=\frac{v^{j-t}(1-v^{t+2})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}},\quad \text{for } j\ge t.$$

The special case $t=0$ appeared already in the previous section:

$$\varphi_j=\frac{v^j(1+v+v^2)}{(1+v)^{j+1}}.$$

Likewise, for $t\ge 1$,

$$\varphi_0=v(1+v+v^2)(1+v)^{t-2}.$$

In particular, the formulæ show that the expression from the previous section can be simplified in general, which could have been seen directly, of course.

Theorem 1. *The generating function of Deutsch path with lower boundary 0 and upper boundary $m-1$, starting at $(0,t)$ and ending at (n,j) is given by*

$$\frac{(1+v)^{t-j-2}(1-v^{j+1})v(1-v^{m-t})(1+v+v^2)}{(1-v)(1-v^{m+2})}, \quad \textit{for } j<t,$$

$$\frac{v^{j-t}(1-v^{t+2})(1-v^{1-j+m})(1+v+v^2)}{(1-v)(1+v)^{j-t+2}(1-v^{m+2})}, \quad \textit{for } j\ge t,$$

with the substitution $z=\dfrac{v}{1+v+v^2}$.

By shifting everything down, we can interpret the results as Deutsch walks between boundaries $-t$ and $m-1-t$, starting at the origin $(0,0)$ and ending at $(n,j-t)$.

Theorem 2. *The generating function of Deutsch path with lower boundary $-t$ and upper boundary h, starting at $(0,0)$ and ending at (n,i) with $-t\le i\le h$ is given by*

$$\frac{(1+v)^{i-2}(1-v^{i+t+1})v(1-v^{h+1})(1+v+v^2)}{(1-v)(1-v^{h+t+3})}, \quad \textit{for } i<0,$$

$$\frac{v^{i}(1-v^{t+2})(1-v^{2-i+h})(1+v+v^2)}{(1-v)(1+v)^{i+2}(1-v^{h+t+3})}, \quad \textit{for } i\ge 0.$$

It is possible to consider the limits $t\to\infty$ and/or $h\to\infty$ resulting in simplified formulæ.

4. Conclusion

Various parameters could be worked out starting from the present findings. Currently, nothing to that effect has been done.

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