# A NOTE ON THE DISTRIBUTION OF THE THREE TYPES OF NODES IN UNIFORM BINARY TREES

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Abstract. We use Zeilberger's algorithm to compute some sums that came up in Mahmoud's analysis of the distribution of types of nodes in binary trees.

In [4], Mahmoud has considered uniform binary trees (counted by Catalan numbers $b_n = \frac{1}{n+1}\binom{2n}{n}$) and the three statistics $X_n^{(0)}$, $X_n^{(1)}$, $X_n^{(2)}$, counting the numbers of internal nodes having 0, 1, 2 internal nodes as successors, respectively. He obtained the following theorem (with one typo removed):

**Theorem 1.** [Mahmoud]

$$\Pr\left\{X_n^{(0)}=j\right\} = \frac{1}{b_n}\sum_{i=0}^{n-j}(-1)^{n-j-i}b_i\binom{i+1}{n-i}\binom{n-i}{j},$$

$$\Pr\left\{X_n^{(1)}=j\right\} = \frac{1}{b_n}\sum_{i=0}^{n-j}(-1)^{n-j-i}2^{n-i}b_i\binom{n-1}{i-1}\binom{n-i}{j},$$

$$\Pr\left\{X_n^{(2)}=j\right\} = \frac{1}{b_n}\sum_{k=0}^{j}(-1)^{j-k}b_k\binom{n-k}{j-k}\sum_{i=0}^{n-k}2^{n-k-i}\binom{n-1}{k+i-1}\binom{k+1}{i}.$$

Mahmoud used generating functions to get these results. For convenience, we sketch how to do it.

The generating function $B(z)$ of the binary trees is normally obtained via the equation $B = 1 + zB^2$. However, we can also get it via the equation $B = 1 + z + 2z(B-1) + z(B-1)^2$. Observe that $B(z)-1$ is the generating function for the nonempty trees, and the recursion now distinguishes between zero, one, or two successors. Now, we can use additional variables $u, v, w$ in order to count nodes with zero, one, or two successors. Then the equation is

$$B = 1 + zu + 2zv(B-1) + zw(B-1)^2,$$

with the solution

$$B(z;u,v,w) = \frac{1-2z(v-w)-\sqrt{1-4zv+4z^2(v^2-uw)}}{2zw}.$$

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Now the probabilities from Theorem 1 are obtained by reading off appropriate coefficients:

$$\Pr\left\{X_n^{(0)}=j\right\}=\frac{1}{b_n}[z^nu^j]B(z;u,1,1)\,,$$
$$\Pr\left\{X_n^{(1)}=j\right\}=\frac{1}{b_n}[z^nv^j]B(z;1,v,1)\,,$$
$$\Pr\left\{X_n^{(2)}=j\right\}=\frac{1}{b_n}[z^nw^j]B(z;1,1,w)\,.$$

With Zeilberger's algorithm EKHAD (see [5]), I found the following explicit formulæ:

**Theorem 2.**

$$\Pr\left\{X_n^{(0)}=j\right\}=\frac{2^{n+1-2j}(n+1)!\,n!\,(n-1)!}{j!\,(j-1)!\,(n+1-2j)!\,(2n)!}\,,$$
$$\Pr\left\{X_{2n+1}^{(1)}=2j\right\}=\frac{2^{2j-1}(2n+2)!\,(2n)!\,(2n)!}{(2j)!\,(n-j+1)!\,(n-j)!\,(4n+1)!}\,,$$
$$\Pr\left\{X_{2n}^{(1)}=2j+1\right\}=\frac{2^{2j+1}(2n+1)!\,(2n)!\,(2n-1)!}{(2j+1)!\,(n-j)!\,(n-j-1)!\,(4n)!}\,,$$
$$\Pr\left\{X_n^{(1)}=j\right\}=0\quad \textit{for } n+j \textit{ even}\,,$$
$$\Pr\left\{X_n^{(2)}=j\right\}=\frac{2^{n-1-2j}(n+1)!\,n!\,(n-1)!}{(j+1)!\,j!\,(n-1-2j)!\,(2n)!}\,.$$

The quantity

$$\Pr\left\{X_n^{(0)}=j\right\}b_n=\frac{2^{n+1-2j}(n-1)!}{j!\,(j-1)!\,(n+1-2j)!}$$

is the number of binary trees of size $n$ and $j$ leaves. It is interesting to compare this quantity with the corresponding one for planted plane trees; the number of planted plane trees of size $n$ with $j$ leaves is the *Narayana* number

$$\frac{1}{n}\binom{n}{j}\binom{n-2}{j-1}=\frac{(n-1)!\,(n-2)!}{j!\,(j-1)!\,(n-1-j)!^2}\,,$$

see [3].

We provide a glimpse of what Zeilberger's algorithm does. Dealing with the first sum in Theorem 1, and setting

$$F(n,i):=\frac{1}{b_n}(-1)^{n-j-i}b_i\binom{i+1}{n-i}\binom{n-i}{j}\,,$$

where we treat $j$ as a parameter, the algorithm computes

$$G(n,i) := \frac{(n+2)(n-2i)(n-2i-1)}{2(n+1-i-j)}\,F(n,i)\,,$$

such that

$$(n-2j+2)(2n+1)\,F(n+1,i) - (n+2)n\,F(n,i) = \\ = G(n,i+1) - G(n,i)\,.$$

Denoting the sum on $i$ by $F(n)$ and summing up, the right hand side telescopes, and we get

$$(n-2j+2)(2n+1)\,F(n+1) = (n+2)n\,F(n)\,.$$

Such a *first order recursion* can always be solved by *iteration*, leading to the announced formula.

If we do the same thing for the second sum, we find the recursion

$$(2n+3)(2n+1)(n-j+3)(n-j+1)\,F(n+2) = \\ = (n+3)(n+2)(n+1)n\,F(n)$$

which is not of first order, but can, because of the lack of the term $F(n+1)$, still be solved by iteration. It turns out that four cases ($n, j$ even or odd) have to be distinguished.

The third entry is a double sum and thus not so easily treatable. I recently showed it to F. Chyzak, and he is able to handle it with his package MGFUN (see [1]). However, because of the following argument, the double sum is reducible to the other instances. We will show that the three statistics are basically the same. Assume that there are $i$ nodes with 0 successors, $j$ nodes with 1 successor, and $k$ nodes with 2 successors. Then we have by an elementary argument the equations

$$\begin{aligned} i+j+k &= n\,,\\ j+2k &= n+1\,, \end{aligned}$$

whence $k = i+1$ and furthermore $j = n-2i-1$. Thus, with

$$H(n,j) := \frac{2^{n+1-2j}(n+1)!\,n!\,(n-1)!}{j!\,(j-1)!\,(n+1-2j)!\,(2n)!}\,,$$

we get alternatively

$$\begin{aligned}
&\Pr\left\{X_n^{(0)}=j\right\}=H(n,j)\,,\\
&\Pr\left\{X_n^{(1)}=j\right\}=H(n,\tfrac{n-j+1}{2}),\qquad \text{for } n+j \text{ is odd}\,,\\
&\Pr\left\{X_n^{(1)}=j\right\}=0,\qquad \text{for } n+j \text{ even}\,,\\
&\Pr\left\{X_n^{(2)}=j\right\}=H(n,j+1)\,.
\end{aligned}$$

The fact, that the covariance matrix has rank 1 (see [4]) becomes perhaps a little bit more transparent by this combinatorial argument.

With Zeilberger's algorithm we can do even more: We can compute the $s$–th factorial moments explicitly. Let $n^{\underline{k}}=n(n-1)\dots(n-k+1)$, and

$$M_{n;s}^{(i)}:=\sum_j \Pr\left\{X_n^{(i)}=j\right\} j^{\underline{s}}$$

denote the $s$–th factorial moments for $i=0,1,2$, then:

**Theorem 3.**

$$\begin{aligned}
M_{n;s}^{(0)}&=\frac{(n+1)!\,n!\,(2n-2s)!}{(2n)!\,(n+1-2s)!\,(n-s)!}\,,\\
M_{n;s}^{(1)}&=\frac{2^s\,(n+1)!\,n!\,(n-1)!\,(2n-2s)!}{(2n)!\,(n-s+1)!\,(n-s)!\,(n-s-1)!}\,,\\
M_{n;s}^{(2)}&=\frac{n!\,(n-1)!\,(2n-2s)!}{(2n)!\,(n-1-2s)!\,(n-s)!}\,.
\end{aligned}$$

We end this little note by citing two other papers ([2], [6]) dealing with binary trees and statistics on the leaves.

## References


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