Two Families of Series for the Generalized Golden Ratio

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Abstract

Higher order Fibonacci numbers have the characteristic equation $X^n - X^n - \cdots - X - 1 = 0$, where $n = 2$ means the classical case. Special interest is in $\alpha = \alpha_n$, the dominant (=largest, positive) root of this equation, which is the golden ratio for $n = 2$.

Let also be $\beta = \beta_n = 1/\alpha_n$, then, as $n \to \infty$, $\alpha_n \to 2$ and $\beta_n \to \frac{1}{2}$. In this paper, series expansions of $(2-\alpha)^r$ and $(\beta - \frac{1}{2})^r$ are obtained for arbitrary exponents r. This extends results recently obtained in [2].

1 Introduction

The generalized Fibonacci sequence of order n is defined by

$$G_i^{(n)} = G_{i-1}^{(n)} + \cdots + G_{i-n}^{(n)}$$

with appropriate initial terms. Here the associated characteristic polynomial is $X^n - X^{n-1} - \cdots - X - 1$. It is well-known [4, 5] that this polynomial has a single positive zero α_n, which is strictly between 1 and 2.

In [2], the following theorem was proven.

Theorem 1 *Let $n \geq 2$, and define $\alpha = \alpha_n$, the positive real zero of $X^n - X^{n-1} - \cdots - X - 1$. Let $\beta = 1/\alpha$. Then*

(a)

$$\beta = \frac{1}{2} + \frac{1}{2}\sum_{k\geq 1}\frac{1}{k}\binom{k(n+1)}{k-1}\frac{1}{2^{k(n+1)}}.$$

(b)

$$\alpha = 2 - 2\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)-2}{k-1}\frac{1}{2^{k(n+1)}}.$$

(c)

$$\frac{1}{2-\alpha} = 2^n - \frac{n}{2} - \frac{1}{2}\sum_{k\geq 1} \frac{1}{k}\binom{k(n+1)}{k+1}\frac{1}{2^{k(n+1)}}.$$

The proof was based on the *Lagrange Inversion Formula.* Here, we use a different method, namely the *Generalized Binomial Series*, as described in [1], where it is also reported that it is due to Lambert in the 1750s [3]. In the book, the results that we need in the next section are proved using Raney's lemma [6], but the Lagrange Inversion Formula could be used to prove them as well. In any case, we are able to streamline the previous analysis by just applying Lambert's identity as well as extend the results by providing expansions for $(2-\alpha)^r$ and $(\beta - \frac{1}{2})^r$ for general real exponents r. Our main new results are the expansions in (1) and (2).

For later reference, let us state that

$$\alpha^n - \frac{\alpha^n - 1}{\alpha - 1} = 0,$$

which readily simplifies to

$$2 - \alpha = \alpha^{-n}.$$

From this we infer

$$2 - \frac{1}{\beta} = \beta^n$$

and in simplified form

$$\beta = \frac{1+\beta^{n+1}}{2}.$$

2 Series expansions

Our starting point is the *generalized binomial series* $\mathcal{B}_t(z)$. We follow the description in [1]: The series is defined by

$$\mathcal{B}_t(z) = \sum_{k\geq 0} \binom{tk+1}{k}\frac{1}{tk+1}z^k,$$

satisfies the equation

$$\mathcal{B}_t(z) = 1 + z\mathcal{B}_t(z)^t,$$

and, for any real number r,

$$\mathcal{B}_t(z)^r = \sum_{k\ge 0} \binom{tk+r}{k} \frac{r}{tk+r} z^k.$$

These definitions and results are due to Lambert [3].

In our instance, $t = n+1$.

As described in the Introduction,

$$\beta = \frac{1+\beta^{n+1}}{2}.$$

Specializing $z = 1/2^{n+1}$, we find that

$$\mathcal{B}_{n+1}\Big(\frac{1}{2^{n+1}}\Big) = 1 + \bigg(\frac{\mathcal{B}_{n+1}\big(\frac{1}{2^{n+1}}\big)}{2}\bigg)^{n+1}.$$

Therefore

$$\beta = \frac{1}{2}\mathcal{B}_{n+1}\Big(\frac{1}{2^{n+1}}\Big),$$

since there is only one positive solution $0 < \beta < 1$, and both, β and $\frac{1}{2}\mathcal{B}_{n+1}\big(\frac{1}{2^{n+1}}\big)$ are positive and satisfy the same equation and are henceforth the same.

Therefore

$$\begin{aligned}\beta &= \frac{1}{2}\sum_{k\ge 0}\binom{(n+1)k+1}{k}\frac{1}{(n+1)k+1}\frac{1}{2^{(n+1)k}}\\ &= \frac{1}{2} + \frac{1}{2}\sum_{k\ge 1}\binom{(n+1)k}{k-1}\frac{1}{k}\frac{1}{2^{(n+1)k}},\end{aligned}$$

which is the expression given before. We also have

$$\beta - \frac{1}{2} = \frac{1}{2}\beta^{n+1},$$

and thus for general r

$$\Big(\beta - \frac{1}{2}\Big)^r = \frac{1}{2^{(n+2)r}}\sum_{k\ge 0}\binom{(n+1)k + r(n+1)}{k}\frac{r}{k+r}\frac{1}{2^{(n+1)k}}. \tag{1}$$

For $r = 1$, this yields

$$\begin{aligned}\beta &= \frac{1}{2} + \frac{1}{2^{n+2}}\sum_{k\ge 0}\binom{(n+1)k+(n+1)}{k}\frac{1}{k+1}\frac{1}{2^{(n+1)k}}\\ &= \frac{1}{2} + \frac{1}{2}\sum_{k\ge 0}\binom{(n+1)(k+1)}{k}\frac{1}{k+1}\frac{1}{2^{(n+1)(k+1)}}\end{aligned}$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{k \ge 1} \binom{(n+1)k}{k-1} \frac{1}{k} \frac{1}{2^{(n+1)k}},$$

which is again the expansion given in [2].

Now we set $\alpha = 1/\beta$ and look for expansions related to it. First,

$$\begin{aligned} \alpha = \beta^{-1} &= 2 \sum_{k \ge 0} \binom{(n+1)k-1}{k} \frac{-1}{(n+1)k-1} \frac{1}{2^{(n+1)k}} \\ &= 2 - 2 \sum_{k \ge 1} \binom{(n+1)k-2}{k-1} \frac{1}{k} \frac{1}{2^{(n+1)k}}, \end{aligned}$$

which is the expansion given in [2].

From

$$\beta - \frac{1}{2} = \frac{1}{2} \beta^{n+1},$$

we have, by multiplication by α and simple rearrangements,

$$2 - \alpha = \beta^n.$$

Thus we have for general r,

$$\begin{aligned} (2-\alpha)^r = \beta^{rn} &= 2^{-rn} \mathcal{B}_{n+1}\Big(\frac{1}{2^{n+1}}\Big)^{rn} \\ &= 2^{-rn} \sum_{k \ge 0} \binom{(n+1)k+rn}{k} \frac{rn}{(n+1)k+rn} \frac{1}{2^{(n+1)k}}. \end{aligned} \tag{2}$$

The instance $r = -1$ was given in [2]:

$$\begin{aligned} \frac{1}{2-\alpha} &= 2^n \sum_{k \ge 0} \binom{(n+1)k-n}{k} \frac{-n}{(n+1)k-n} \frac{1}{2^{(n+1)k}} \\ &= 2^n - \frac{n}{2} + \sum_{k \ge 2} \binom{(n+1)k-n}{k} \frac{-n}{(n+1)k-n} \frac{1}{2^{(n+1)k-n}} \\ &= 2^n - \frac{n}{2} + \sum_{k \ge 1} \binom{(n+1)k+1}{k+1} \frac{-n}{(n+1)k+1} \frac{1}{2^{(n+1)k+1}} \\ &= 2^n - \frac{n}{2} - \frac{1}{2} \sum_{k \ge 1} \binom{(n+1)k}{k+1} \frac{n}{nk} \frac{1}{2^{(n+1)k}} \\ &= 2^n - \frac{n}{2} - \frac{1}{2} \sum_{k \ge 1} \binom{(n+1)k}{k+1} \frac{1}{k} \frac{1}{2^{(n+1)k}}, \end{aligned}$$

as predicted.

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