THE ORTHOGONALITY OF GEGENBAUER POLYNOMIALS $C_n^\lambda(x)$ FOR $\lambda < -n$

KATHY DRIVER AND HELMUT PRODINGER

The Gegenbauer (or ultraspherical) polynomials $C_n^\lambda(z)$ can be defined by the generating function

$$(1-2zr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(z)\, r^n;$$

see [1]. For $\lambda > -1/2$, these polynomials $C_n^\lambda(z)$ are *orthogonal* on the interval $(-1,1)$ with respect to the *weight function* $(1-z^2)^{\lambda-1/2}$ and therefore have all of their zeros in that interval. As the parameter λ decreases below $-1/2$, the zeros of $C_n^\lambda(z)$ depart from $(-1,1)$ and when $\lambda < 1-n$, all the zeros of $C_n^\lambda(z)$ lie on the imaginary axis (cf. [2]).

An interesting question is whether there is some underlying orthogonality that gives rise to the location of the zeros of $C_n^\lambda(z)$ on the imaginary axis for $\lambda < 1-n$. We show that this is indeed the case for $\lambda < -n$ by proving that the sequence of polynomials $\{i^n C_n^\lambda(ix)\}_{n=1}^{\infty}$ is orthogonal on $(-\infty,\infty)$ with respect to the weight function $(1+x^2)^{\lambda-1/2}$. Note that for a given parameter λ, the set of polynomials is finite ($1 \le n < -\lambda$). This orthogonality property does not appear to hold for $-n < \lambda < 1-n$. Since $C_n^\lambda(z)$ is an even/odd function of z according to whether n is even/odd, the sequence of polynomials $\{i^n C_n^\lambda(ix)\}$ is real–valued for x real, while the weight function $(1+x^2)^{\lambda-1/2}$ is positive for all x. It therefore follows from the classical theory of real orthogonal polynomials that all zeros of $C_n^\lambda(ix)$ are real for $\lambda < -n$. Thus the zeros of $C_n^\lambda(z)$ are pure imaginary for $\lambda < -n$.

As an application of the orthogonality of $\{i^n C_n^\lambda(ix)\}$ for $\lambda < -n$, we derive an expression for

$$\int_{-\infty}^{\infty} C_l^\lambda(ix)\, C_m^\lambda(ix)\, C_n^\lambda(ix)\, (1+x^2)^{\lambda-1/2}\, dx$$

which is analogous to the result given in [1, p. 321, eq. (6.8.10)] for $\lambda > -1/2$.

Theorem 1. *For $n, m < -\lambda$ we have*

$$\int_{-\infty}^{\infty} i^n\, C_n^\lambda(ix)\, i^m\, C_m^\lambda(ix)\ (1+x^2)^{\lambda-1/2}\, dx = g_n \delta_{nm} \tag{1}$$

with

$$g_n = \frac{(-1)^n 2^{-2\lambda-1}\big[(\lambda)_n\big]^2\big[\Gamma(-n-\lambda)\big]^2}{n!\,(n+2\lambda)_n\,\Gamma(-2n-2\lambda)}. \tag{2}$$

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Proof. The *Rodrigues formula* for the Gegenbauer polynomials is given by (cf. [1, p. 303, eq. (6.4.14)])

$$(1-x^2)^{\lambda-1/2}C_n^\lambda(x)=\frac{(-2)^n(\lambda)_n}{n!\,(n+2\lambda)_n}\,D^n\left[(1-x^2)^{n+\lambda-1/2}\right]. \tag{3}$$

Replacing x by ix yields

$$(1+x^2)^{\lambda-1/2}C_n^\lambda(ix)=\frac{(-2)^n(\lambda)_n}{n!\,(n+2\lambda)_n\,i^n}\,D^n\left[(1+x^2)^{n+\lambda-1/2}\right]. \tag{4}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} & i^n\,C_n^\lambda(ix)\,i^m\,C_m^\lambda(ix)\,\left(1+x^2\right)^{\lambda-1/2}dx \\ &= \frac{(-2)^n(\lambda)_n}{n!\,(n+2\lambda)_n}\int_{-\infty}^{\infty}\left\{D^n\left[(1+x^2)^{n+\lambda-1/2}\right]\right\}i^m\,C_m^\lambda(ix)\,dx.\end{aligned} \tag{5}$$

Integrating the right hand side of (5) by parts n times, each time differentiating $i^mC_m^\lambda(ix)$ and integrating the expression in curly brackets, we observe that at the k^{th} stage, the integrated part $\pm\{D^{n-k}(1+x^2)^{n+\lambda-1/2}\}D^{k-1}(i^mC_m^\lambda(ix))$ has limit value zero as x tends to $\pm\infty$, since $\lambda<-n$. After n such integrations by parts we therefore have

$$\begin{aligned}\int_{-\infty}^{\infty} & i^n\,C_n^\lambda(ix)\,i^m\,C_m^\lambda(ix)\,\left(1+x^2\right)^{\lambda-1/2}dx \\ &= \frac{(-1)^n(-2)^n(\lambda)_n}{n!\,(n+2\lambda)_n}\int_{-\infty}^{\infty}(1+x^2)^{n+\lambda-1/2}\,D^n\big(i^m\,C_m^\lambda(ix)\big)\,dx.\end{aligned} \tag{6}$$

If $m\neq n$, we may choose n to be larger (or interchange m and n in the preceding argument) and conclude that

$$\int_{-\infty}^{\infty} i^n\,C_n^\lambda(ix)\,i^m\,C_m^\lambda(ix)\,\left(1+x^2\right)^{\lambda-1/2}dx=0,$$

because $C_m^\lambda(ix)$ is a polynomial of degree m.

For $m=n$, the right hand side of (6) is equal to

$$\frac{2^n(\lambda)_n}{n!\,(n+2\lambda)_n}\int_{-\infty}^{\infty}(1+x^2)^{n+\lambda-1/2}\,D^n\big(i^n\,C_n^\lambda(ix)\big)\,dx=g_n.$$

Now (cf. [3, p. 277])

$$D^n\left[C_n^\lambda(x)\right]=2^n(\lambda)_n,$$

so it follows that

$$g_n=\frac{(-1)^n2^{2n}\left[(\lambda)_n\right]^2}{n!\,(n+2\lambda)_n}\int_{-\infty}^{\infty}(1+x^2)^{n+\lambda-1/2}\,dx. \tag{7}$$

To evaluate the integral, we use the *beta integral on a halfline*, ([1, p. 7, eq. (1.1.20)])

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,ds = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}\ :$$

$$\begin{aligned}\int_{-\infty}^{\infty}(1+x^2)^{n+\lambda-1/2}\,dx &= 2\int_0^\infty (1+x^2)^{n+\lambda-1/2}\,dx\\ &= \int_0^\infty (1+t)^{n+\lambda-1/2}\,t^{-1/2}\,dt = \frac{\sqrt{\pi}\,\Gamma(-n-\lambda)}{\Gamma(-n-\lambda+\frac{1}{2})}.\end{aligned}$$

From this the formula for g_n is immediate using the duplication formula for the Gamma function. ☐

Corollary. For $l, m, n < -\lambda$

$$\begin{aligned}&\int_{-\infty}^{\infty} C_l^\lambda(ix)\,C_m^\lambda(ix)\,C_n^\lambda(ix)\,(1+x^2)^{\lambda-1/2}\,dx\\ &\qquad = \frac{(-1)^s 2^{-2\lambda-1}\,(\lambda)_{s-l}\,(\lambda)_{s-n}\,(\lambda)_{s-m}\,(2\lambda)_s\,(\lambda)_l\,(\lambda)_{l+1}\big[\Gamma(-l-\lambda)\big]^2}{(s+\lambda)\,(s-l)!\,(s-n)!\,(s-m)!\,(\lambda)_s\,(2\lambda)_{2l}\,\Gamma(-2l-2\lambda)}\end{aligned}$$

where $l+m+n=2s$ is even and the sum of any two of l, m, n is not less than the third. The integral is zero is all other cases.

Proof. The *linearization formula* for the product of two Gegenbauer polynomials is given by (cf. [1, p. 319, eq. (6.8.4)])

$$C_m^\lambda(x)\,C_n^\lambda(x) = \sum_{k=0}^{\min(m,n)} a(k,m,n)\,C_{m+n-2k}^\lambda(x) \tag{8}$$

where

$$a(k,m,n) = \frac{(m+n+\lambda-2k)\,(\lambda)_k\,(\lambda)_{m-k}\,(\lambda)_{n-k}\,(2\lambda)_{m+n-k}\,(m+n-2k)!}{(m+n+\lambda-k)\,k!\,(m-k)!\,(n-k)!\,(\lambda)_{m+n-k}\,(2\lambda)_{m+n-2k}}.$$

From (8) it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty} i^l\,C_l^\lambda(ix)\,i^m\,C_m^\lambda(ix)\,i^n\,C_n^\lambda(ix)\,(1+x^2)^{\lambda-1/2}\,dx\\ &\qquad = \sum_{k=0}^{\min(m,n)} a(k,m,n)\int_{-\infty}^{\infty} i^{m+n}\,C_{m+n-2k}^\lambda(ix)\,i^l\,C_l^\lambda(ix)\,(1+x^2)^{\lambda-1/2}\,dx\end{aligned}\tag{9}$$

where by Theorem 1 the integral on the right hand side of (9) is zero for $l \neq m+n-2k$. Hence only the term with $k = s-l = m+n-s$ survives in the sum, and the right hand side of (9) reduces to

$$\begin{aligned}&a(m+n-s,m,n)\,i^{m+n-l}\int_{-\infty}^{\infty}\Big[i^l\,C_l^\lambda(ix)\Big]^2\,(1+x^2)^{\lambda-1/2}\,dx\\ &\quad = a(m+n-s,m,n)(-1)^{s-l}g_l\\ &\quad = \frac{(l+\lambda)\,(\lambda)_{s-l}\,(\lambda)_{s-n}\,(\lambda)_{s-m}\,(2\lambda)_s\,l!}{(s+\lambda)\,(s-l)!\,(s-n)!\,(s-m)!\,(\lambda)_s\,(2\lambda)_l}\,\frac{(-1)^s 2^{-2\lambda-1}\big[(\lambda)_l\big]^2\big[\Gamma(-l-\lambda)\big]^2}{l!\,(l+2\lambda)_l\,\Gamma(-2l-2\lambda)},\end{aligned}$$

where we have used (1). This is equivalent to the formula given in the corollary. $\square$

References

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Kathy Driver, Helmut Prodinger, The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa, email: 036kad@cosmos.wits.ac.za, helmut@gauss.cam.wits.ac.za.