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# FIBONACCI NUMBERS OF GRAPHS: II

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## 1. INTRODUCTION

In [9], the Fibonacci number $f(G)$ of a (simple) graph $G$ is introduced as the total number of all Fibonacci subsets $S$ of the vertex set $V(G)$ of $G$, where a Fibonacci subset $S$ is a (possibly empty) subset of $V(G)$ such that any two vertices of $S$ are not adjacent. In *Graph Theory* [6, p. 257] a Fibonacci subset is called an independent set of vertices. From [9] we have the elementary inequality

$$F_{n+1} \leqslant f(G) \leqslant 2^{n-1} + 1, \tag{1.1}$$

where $F_n$ denotes the usual Fibonacci numbers with

$$F_0 = F_1 = 1,\ F_n = F_{n-1} + F_{n-2},$$

and $G$ is a tree with $n$ vertices. Furthermore, several problems are formulated concerning the Fibonacci numbers of some special graphs. The present aim is to derive a formula for $f(T_n(t))$, where $T_n(t)$ is the full $t$-ary tree with height $n$: [$T_0(t)$ is the empty tree.]

1 vertex ... ... height 1
$t$ vertices ... ... height 2
$t^2$ vertices ... ... height 3
$t^{n-2}$ vertices ... ... height $n - 1$
$t^{n-1}$ vertices ... ... height $n$

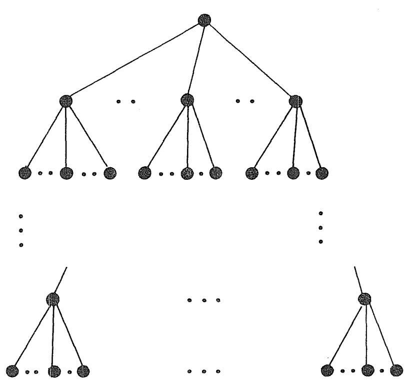


FIGURE 1

For $t = 1$, one can see immediately that $f(T_n(t)) = F_{n+1}$, so the interesting cases are $t \geqslant 2$. In Section 2, for $t = 2, 3, 4$, the asymptotic formula

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \qquad (n \to \infty) \tag{1.2}$$

is proved, where $A(t)$ and $k(t)$ denote constants (depending on $t$) with

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)}.$$

In Section 3, it is proved that for $t \geqslant 5$ such an asymptotic formula does not hold; we show that for $t \geqslant 5$:

$$f(T_{2m}(t)) \sim B(t) \cdot k(t)^{t^{2m}}$$
$$f(T_{2m+1}(t)) \sim C(t) \cdot k(t)^{t^{2m+1}} \tag{1.3}$$

where $B(t) > C(t)$ are constants depending on $t$ with

$$\lim_{t\to\infty} B(t) = \lim_{t\to\infty} C(t) = 1.$$

In Section 4, we establish an asymptotic formula for the average value $S_n$ of the Fibonacci number of binary trees with $n$ vertices (where all such trees are regarded equally likely). For the sake of brevity, we restrict our considerations to the important case of binary trees; however, the methods would even be applicable in the very general case of so-called "simply generated families of trees" introduced by Meir and Moon [8].

By a version of Darboux's method (see Bender's survey [1]), we derive

$$S_n \sim G \cdot r^n \qquad (n \to \infty), \tag{1.4}$$

where $G = 1,\ 12928\ldots$ and $r = 1,\ 63742\ldots$ are numerical constants.

## 2. FIBONACCI NUMBERS OF $t$-ARY TREES ($t = 2, 3, 4$)

By a simple argument (compare [9]), the following recursion holds for the Fibonacci number $x_n\colon = f(T_n(t))$,

$$x_{n+1} = x_n^t + x_{n-1}^{t^2} \text{ with } x_0 = 1,\ x_1 = 2. \tag{2.1}$$

We proceed as in [4] and put $y_n = \log x_n$; by (2.1),

$$y_{n+1} = ty_n + \alpha_n \text{ with } \alpha_n = \log\left(1 + \frac{x_{n-1}^{t^2}}{x_n^t}\right). \tag{2.2}$$

Because of

$$x_{n-1}^{t^2} < (x_{n-1}^t + x_{n-2}^{t^2})^t = x_n^t,$$

the estimate

$$0 < \alpha_n < \log 2 \tag{2.3}$$

results. The solution of recursion (2.2) is given by

$$y_n = t^n\left(\frac{\alpha_0}{t} + \frac{\alpha_1}{t^2} + \cdots + \frac{\alpha_{n-1}}{t^n}\right).$$

It is now convenient to extend the series in $\alpha_i$ to infinity [because of (2.3) the series is convergent]:

$$Y_n: = \sum_{i=0}^{\infty} t^{n-1-i}\alpha_i. \tag{2.4}$$

For the difference

$$r_n: = Y_n - y_n = \sum_{i=n}^{\infty} t^{n-1-i}\alpha_i,$$

we have

$$0 < r_n \leqslant \frac{\log 2}{t-1}. \tag{2.5}$$

Therefore, we obtain

$$x_n = e^{Y_n - r_n} = e^{-r_n} \cdot k(t)^{t^n}, \tag{2.6}$$

where

$$k(t) = \exp\left(\sum_{i=0}^{\infty} t^{-i-1}\alpha_i\right) \tag{2.7}$$

and $1 < k(t) < 2^{1/(t-1)}$ by (2.3).

In the following, we investigate the factor $e^{-r_n}$ of (2.6); we set

$$q_n = x_n^t / x_{n+1}$$

and obtain the recursion

$$q_{n+1} = \frac{1}{1 + q_n^t},\ q_0 = \frac{1}{2} \tag{2.8}$$

from (2.1). It is useful to split up the sequence $(q_n)$ into two complementary subsequences

$$\begin{aligned} (g_m): &= (q_{2m}) = (q_0, q_2, \ldots), \\ (u_m): &= (q_{2m+1}) = (q_1, q_3, \ldots). \end{aligned} \tag{2.9}$$

Lemma 1

The following inequalities hold for the subsequences $(g_m)$ and $(u_m)$ of $(q_n)$:

(i) $g_{m+1} > g_m$ for all $m = 0, 1, 2, 3, \ldots$

(ii) $u_{m+1} < u_m$ for all $m = 0, 1, 2, 3, \ldots$

(iii) $u_m > g_m$ for all $m = 0, 1, 2, 3, \ldots$ .

Proof: Let $q_{n-2} > q_n$; then $1 + q_{n-2}^t > 1 + q_n^t$, $1/(1 + q_{n-2}^t) < 1/(1 + q_n^t)$, and so

$$\frac{1}{1 + \left(\dfrac{1}{1 + q_{n-2}^t}\right)^t} > \frac{1}{1 + \left(\dfrac{1}{1 + q_n^t}\right)^t}.$$

Applying (2.8), we have proved:

$$\text{If } q_{n-2} > q_n, \text{ then } q_n > q_{n+2}. \tag{2.10}$$

Because of $g_1 > g_0$, (i) is proved by induction; (ii) and (iii) follow by a similar argument.

By Lemma 1, $(g_m)$ and $(u_m)$ are monotone sequences with the obvious bounds

$$\frac{1}{2} \leqslant g_m < u_m \leqslant 1. \tag{2.11}$$

So the sequences $(g_m)$ and $(u_m)$ must be convergent to limits $g$ and $u$ (depending on $t$). The following proposition shows that $g = u$ in the cases $t = 2, 3, 4$.

Proposition 1

For $t = 2, 3, 4$, the sequence $(q_n)$ is convergent to a limit $w(t)$, where $w(t)$ is the unique root of the equation $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} \leqslant w(t) \leqslant 1$.

Proof: By Lemma 1 we only have to show that $(g_m)$ and $(u_m)$ are convergent to the same limit. For $(g_m)$ and $(u_m)$ the following system of recursions holds:

$$\begin{aligned} u_m &= \frac{1}{1 + g_m^t} \\ g_{m+1} &= \frac{1}{1 + u_m^t}. \end{aligned} \tag{2.12}$$

Taking the limit $m \to \infty$, we obtain

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \text{ with } \frac{1}{2} \leqslant g \leqslant u \leqslant 1. \tag{2.13}$$

Let us start with the case $t = 2$. By (2.13), we have $ug^2 = 1 - u$, $gu^2 = 1 - g$, and therefore, $u - u^2 = g - g^2$. Because the function $x \mapsto x - x^2$ is strictly decreasing in the interval $\left[\frac{1}{2}, 1\right]$, $u = g$ follows immediately.

In the case $t = 3$, we derive in a similar way the relation $u^2 - u^3 = g^2 - g^3$. Since the function $x \mapsto x^2 - x^3$ is strictly decreasing in the interval $\left[\frac{2}{3}, 1\right]$ and $\frac{2}{3} < g_4 = 0,684\ldots$, we obtain $u = g$ again.

Since the function $x \mapsto x^3 - x^4$ is strictly decreasing in the interval $\left[\frac{3}{4}, 1\right]$ and $g_{73} = 0,7500138\ldots > \frac{3}{4}$, we obtain $u = g$ in the case $t = 4$, too.

So $u = g$ in all considered cases; therefore, a limit $w(t)$ of $(q_n)$ exists for $t = 2, 3, 4$, and $w(t)$ fulfills the equation

$$w = \frac{1}{1 + w^t}.$$

Since the function $f(w) = w^{t+1} + w - 1$ is strictly monotone in the interval $\left[\frac{1}{2}, 1\right]$ and $f\left(\frac{1}{2}\right) < 0$, $f(1) > 0$, there exists a unique root of this equation in the interval $\left[\frac{1}{2}, 1\right]$, which is the limit $w(t)$ from above.

By (2.2) we derive

$$\lim_{n \to \infty} \alpha_n = \log(1 + w(t)^t). \tag{2.14}$$

Because of

$$\left| r_n - \frac{1}{t - 1} \log(1 + w(t)^t) \right| \leqslant \sum_{i=n}^{\infty} t^{n-1-i} |\alpha_i - \log(1 + w(t)^t)| < \frac{\varepsilon}{t - 1}$$

[for all $\varepsilon > 0$, $n \geqslant n_0(\varepsilon)$], the sequence $(r_n)$ is convergent; so

$$\lim_{n \to \infty} e^{-r_n} = (1 + w(t)^t)^{-1/(t-1)} = w(t)^{1/(t-1)} = (1 - w(t))^{1/(t^2-1)} \tag{2.15}$$

results. Altogether we have established:

Theorem 1

Let $T_n(t)$ be the full $t$-ary tree ($t = 2, 3$, or $4$) with height $n$. Then, the Fibonacci number $f(T_n(t))$ fulfills the following asymptotic formula:

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \qquad (n \to \infty)$$

where $A(t) = w(t)^{1/(t-1)}$ and $k(t)$, defined by (2.7), are constants (only depending on $t$) bounded by

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)};$$

$w(t)$ is the unique root of $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} < w(t) < 1$.

Remark: The numerical values of $w(t)$ are

$$w(2) = 0,68233\ldots,\; w(3) = 0,72449\ldots, \text{ and } w(4) = 0,75488\ldots .$$

## 3. FIBONACCI NUMBERS OF $t$-ARY TREES ($t \geqslant 5$)

In this section we consider $t$-ary trees with $t \geqslant 5$. Let $(g_m)$, $(u_m)$ be the subsequences of $(q_n)$ defined by (2.9). $(g_m)$ and $(u_m)$ are convergent to limits $g$ and $u$, respectively (depending on $t$). We shall prove that $g \neq u$; therefore, $(q_n)$ has two accumulation points. For $g$, $u$ the following system of equations holds,

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \tag{3.1}$$

and $g = u$ if and only if $u$ or $g$ is the unique solution of

$$w^{t+1} + w - 1 = 0 \tag{3.2}$$

in the interval $\left[\frac{1}{2}, 1\right]$. If $(u', g')$ and $(u'', g'')$ are two pairs fulfilling (3.1) with $u' < u''$, then $g' > g''$. Let $(\bar{u}, \bar{g})$ denote the pair of solutions with minimal $g$ and maximal $u$.

Lemma 2

The subsequence $(g_m)$ of $(q_n)$ is convergent to the limit $\bar{g}$ and the subsequence $(u_m)$ to the limit $\bar{u}$.

Proof: First we show that $g_m < \bar{g}$ implies $u_m > \bar{u}$ and $g_{m+1} < \bar{g}$.

Because of $g_m < \bar{g}$, we obtain $1 + g_m^t < 1 + \bar{g}^t = 1/\bar{u}$, and so $u_m > \bar{u}$. From $u_m > \bar{u}$, it follows that $1 + u_m^t > 1 + \bar{u}^t = 1/\bar{g}$, hence $g_{m+1} < \bar{g}$.

Using the fact that $g_0 = \frac{1}{2} < \bar{g}$, we obtain, by induction,

$$\lim_{m\to\infty} g_m \leqslant \bar{g} \quad \text{and} \quad \lim_{m\to\infty} u_m \geqslant \bar{u}.$$

By the definition of $(\bar{u}, \bar{g})$, it follows that

$$\lim_{m\to\infty} g_m = \bar{g} \quad \text{and} \quad \lim_{m\to\infty} u_m = \bar{u},$$

and the Lemma is proved.

Lemma 3

Let $t \geqslant 5$ be a positive integer; then there exists a solution $(u, g)$ of the system (3.1) with

$$\frac{1}{2} < g < \frac{1}{2} + \frac{1}{t}.$$

Proof: System (3.1) is equivalent to the equation

$$g = \cfrac{1}{1 + \cfrac{1}{(1 + g^t)^t}} \tag{3.3}$$

We consider the function

$$\varphi_t(g) = \frac{(1 + g^t)^t}{1 + (1 + g^t)^t} - g,$$

and obtain $\varphi_t\left(\frac{1}{2}\right) > 0$; in the sequel, we show $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$. For $t = 5$ or $6$, this inequality can be shown by direct computation:

$$\varphi_5\left(\frac{7}{10}\right) = -0,\ 01502 \quad \text{and} \quad \varphi_6\left(\frac{2}{3}\right) = -0,\ 04306.$$

Let us assume $t \geqslant 7$ in the sequel. By elementary manipulations, the inequality

$$\frac{1}{1 + \frac{1}{\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t}} - \frac{1}{2} - \frac{1}{t} < 0$$

is equivalent to $1 + \left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^{-t} > \frac{2t}{t + 2}$ or

$$\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t < \frac{t + 2}{t - 2}. \tag{3.4}$$

Because of $\left(1 + \frac{2}{t}\right)^{t/2} < e$, it is sufficient to prove

$$\left(1 + \frac{e^2}{2^t}\right)^t < \frac{t + 2}{t - 2}: \tag{3.5}$$

We have

$$\left(1 + \frac{e^2}{2^t}\right)^t \leqslant \exp\left(\frac{e^{2t}}{2^t}\right)$$

and

$$\exp\left(\frac{e^{2t}}{2^t}\right) \leqslant \frac{t + 2}{t - 2} \text{ for } t \geqslant 7.$$

So $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$, and the Lemma is proved, because the continuous function $\varphi$ has a root between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{t}$.

Equation (3.3) is equivalent to

$$g(g^t + 1)^t - (g^t + 1)^t + g = 0. \tag{3.6}$$

The polynomial on the left-hand side of (3.6) is divisible by $g^{t+1} + g - 1$. Because of $\left(\frac{3}{4}\right)^{t+1} + \frac{3}{4} - 1 < 0$ (for $t \geqslant 5$), the unique solution $w(t)$ of $g^{t+1} + g - 1 = 0$ is contained in the interval $\left[\frac{3}{4}, 1\right]$. By Lemma 3, we have found a pair of solutions $(u, g)$ with $u \neq g$ such that $\frac{1}{2} < g < \frac{3}{4} < w(t) < u < 1$. We denote by $(u(t), g(t))$, $t \geqslant 5$, the pair of solutions of (3.1) such that $g(t)$ is minimal and $u(t)$ is maximal. Because of $g(t) < \frac{1}{2} + \frac{1}{t}$ and $u(t) = 1 + g(t)^{-t}$ for $t \geqslant 5$, we obtain

$$\lim_{t \to \infty} g(t) = \frac{1}{2}, \quad \lim_{t \to \infty} u(t) = 1. \tag{3.7}$$

Altogether, we have proved:

Theorem 2

Let $T_n(t)$ be the full $t$-ary tree (for $t \geqslant 5$) with height $n$. Then the Fibonacci numbers fulfill the following asymptotic formulas, respectively:

$$f(T_{2m}(t)) \sim B(t) \cdot k(t)^{t^{2m}},$$

$$f(T_{2m+1}(t)) \sim C(t) \cdot k(t)^{t^{2m+1}},$$

where

$$C(t) = (g(t)^t u(t))^{1/(t^2-1)} = (1 - u(t))^{1/(t^2-1)},$$

$$B(t) = (g(t)u(t)^t)^{1/(t^2-1)} = (1 - g(t))^{1/(t^2-1)},$$

and $k(t)$, defined by (2.7), are constants (only depending on $t$) bounded by

$$2^{1/(1-t)} < C(t) < B(t) < 1 < k(t) < 2^{1/(t-1)};$$

$g(t)$ is the minimal root and $u(t)$ the maximal root of

$$x(x^t + 1)^t - (x^t + 1)^t + x = 0$$

in the interval $\left[\frac{1}{2}, 1\right]$; furthermore,

$$\lim_{t\to\infty} B(t) = \lim_{t\to\infty} C(t) = 1.$$

Remark: In [2], similar recurrences are treated by a slightly different method. The recursion for $(q_n)$ can be considered as a fixed-point problem and our results can be derived in principal by studying this fixed-point problem.

## 4. THE AVERAGE FIBONACCI NUMBER OF BINARY TREES

The family $\beta$ of all binary trees is defined by the following formal equation ($\square$ is the sumbol for a leaf and $\circ$ for an internal node):

$$\beta = \square + \begin{array}{ccc} & \circ & \\ / & & \backslash \\ \beta & & \beta \end{array}; \tag{4.1}$$

(this notation is due to Ph. Flajolet [3]). The generating function

$$B(z) = \sum_{n \geqslant 0} b_n z^n$$

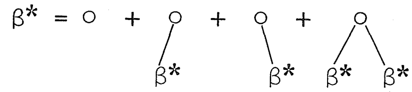
of the numbers of binary trees with $n$ internal nodes is given by

$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \tag{4.2}$$

and, therefore,

$$b_n = \frac{1}{n+1}\binom{2n}{n}. \tag{4.3}$$

For technical reasons, we consider the family $\beta^*$ of all binary trees with leaves removed; $\beta^*$ fulfills



(4.4)

Let $\beta_n$ be the family of binary trees $t$ with $n$ internal nodes, and let

$$f(z) = \sum_{n\geqslant 1} f_n z^n \quad \text{and} \quad g(z) = \sum_{n\geqslant 1} g_n z^n$$

be generating functions of

$$\begin{aligned} f_n &= \sum_{T\in\beta_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset } \textit{not} \text{ containing the root}\}, \\ g_n &= \sum_{T\in\beta_n} \operatorname{card}\{S : S \subseteq V(T);\ S \text{ a Fibonacci subset containing the root}\}. \end{aligned} \tag{4.5}$$

Obviously, the average value of the Fibonacci number of a binary tree with $n$ internal nodes is given by

$$S_n = \frac{h_n}{b_n} \quad \text{with} \quad h_n = f_n + g_n. \tag{4.6}$$

The remainder of this paper is devoted to the asymptotic evaluation of $S_n$. By Stirling's approximation of the factorials, the well-known formula

$$b_n \sim \frac{1}{\sqrt{\pi}}\, 2^{2n} n^{-3/2} \qquad (n \to \infty) \tag{4.7}$$

holds and we can restrict our attention to $h_n$.

For the generating functions, we obtain

$$\begin{aligned} f &= z + z(f + g) + z(f + g) + z(f + g)^2 \\ g &= z + zf + zf + zf^2. \end{aligned} \tag{4.8}$$

[The contributions of (4.8) correspond to the terms in (4.4).] Setting

$$y(z) = 1 + f(z) + g(z),$$

we derive, by some elementary manipulations,

$$z^3y^4 + (2z^2 + z)y^2 - y + (z + 1) = 0. \tag{4.9}$$

Now we want to apply Theorem 5 of [1]; for this purpose, we have to determine the singularity $\rho$ of $y(z)$ nearest to the origin. (4.9) is an implicit representation of $y(z)$. Abbreviating the left-hand side of (4.9) by $F(z, y)$, the singularity $\rho$ (nearest to the origin) and $\sigma = y(\rho)$ are given as solutions of the following system of algebraic equations:

$$F(z,\ y) = 0,$$
$$\frac{\partial F}{\partial y}(z,\ y) = 0. \tag{4.10}$$

Now $\rho$ and $\sigma$ are simple roots of the above equations. By a theorem of Pringsheim [7, p. 389], $\rho$ and $\sigma$ are positive (real) numbers. Using the two-dimensional version of Newton's algorithm (starting with $z_0 = 0,\ 2$ and $y_0 = 1$), we obtain the following numerical values:

$$\rho = 0,\ 15268\ldots \quad \text{and} \quad \sigma = 2,\ 15254\ldots\ . \tag{4.11}$$

Now Theorem 5 of [1] allows us to formulate the following:

Proposition 2

$$\begin{aligned} h_n &\sim \left(\frac{\rho \cdot F_z(\rho,\ \sigma)}{2\pi \cdot F_{yy}(\rho,\ \sigma)}\right)^{1/2} \cdot \rho^{-n} \cdot n^{-3/2}, \qquad (n \to \infty) \\ &\sim (0,\ 63713\ldots) \quad (0,\ 15268\ldots)^{-n} \cdot n^{-3/2}. \end{aligned} \tag{4.12}$$

Altogether, we have proved:

Theorem 3

The average value $S_n$ of the Fibonacci number of a binary tree with $n$ internal nodes fulfills asymptotically

$$S_n \sim G \cdot r^n \qquad (n \to \infty),$$

where $G = 1,\ 12928\ldots$ and $r = 1,\ 63742\ldots$ are numerical constants.

## ACKNOWLEDGMENT

We would like to express our thanks for helpful discussions to Professor G. Baron and Dr. Ch. Buchta.

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# CORRIGENDA TO "SOME SEQUENCES LIKE FIBONACCI'S"

B. H. NEUMANN and L. G. WILSON

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The following changes should be made in the above article. These errors are the responsibility of the editorial staff and were recently brought to the editor's attention by the authors.

p. 80, at the end of formula (1), add superscript "$n$".
p. 81, in formula (7), replace the second "$y$" by "$t$".
p. 82, in the line following (8), add subscript "$d$" to the last "$a$".
p. 82, in the line following (10), add subscript "$d$" to the last "$\alpha$".
p. 83, line 3, insert "growth" between "slower" and "rate".
p. 83, end of text and reference, delete "t" from the name "Johnson".

Gerald E. Bergum

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