THE BOX PARAMETER FOR WORDS AND PERMUTATIONS

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ABSTRACT. The box parameter for words counts how often two letters w_j and w_k define a "box" such that all the letters $w_{j+1}, \dots, w_{k-1}$ fall into that box. It is related to the visibility parameter and other parameters on words. Three models are considered: Words over a finite alphabet, permutations, and words with letters following a geometric distribution. A typical result is: The average box parameter for words over an M letter alphabet is asymptotically given by $2n - 2nH_M/M$, for fixed M and $n \to \infty$.

1. INTRODUCTION

Consider a word $w_1 \dots w_n$ of length n where the letters are positive integers. The visibility parameter (=the number of visible pairs) is defined as

$$\mathsf{Vis}(w_1 \dots w_n) = \#\{(j,k) \mid 1 \le j < k \le n,\ w_l > \max\{w_j, w_k\} \text{ for all } j < l < k\}.$$

Using indicator variables $\chi_{j,k}$, defined by

$$\chi_{j,k}(w_1 \dots w_n) = \begin{cases} 1 & \text{if } w_l > \max\{w_j, w_k\} \text{ for all } j < l < k, \\ 0 & \text{otherwise,} \end{cases}$$

we can write

$$\mathsf{Vis} = \sum_{1 \le j < k \le n} \chi_{j,k}.$$

Gutin et al. [4] have investigated this parameter motivated by horizontal visibility graphs (HVG), which provide a method for studying time series by investigating graphs associated to them. This analysis was largely extended by Cristea and Prodinger [2].

In the last paper, it was also pointed out that there are similar parameters already in the literature; they have some relevance in Computer Science:

The first one is "Knuth's parameter a" (which might also be called left-sided pathlength) [5]: it is defined as

$$\mathsf{a}(w_1 \dots w_n) = \#\{(j,k) \mid 1 \le j < k \le n,\ w_j < w_l \text{ for all } j < l \le k\}.$$

The other one [6] is related to the pathlength in binary search trees:

$$\rho(w_1 \dots w_n) = \#\{(j,k) \mid 1 \le j < k \le n,\\ w_j = \min\{w_j, \dots, w_k\} \text{ or } w_k = \min\{w_j, \dots, w_k\}\}.$$

Date: September 12, 2012.
2010 *Mathematics Subject Classification.* 05A05, 05A30.
Key words and phrases. Words, permutations, q-enumeration.

The purpose of this paper is to study another parameter that we call the *box parameter*. It has some similarity with the above mentioned parameters and is natural from a combinatorial/geometrical point of view. It counts how often the elements between two letters in the word "live in the box defined by them". To make this more precise, let us use indicator variables $\chi_{j,k}$, defined by

$$\chi_{j,k}(w_1\dots w_n)=\begin{cases}1 & \text{if } \min\{w_j,w_k\}<w_l<\max\{w_j,w_k\} \text{ for all } j<l<k,\\ 0 & \text{otherwise,}\end{cases}$$

then we can write

$$\mathsf{Box}=\sum_{1\le j<k\le n}\chi_{j,k}.$$

The box parameter is computationally more involved than the visibility parameter. We compute expectation and variance for the model of words over a finite alphabet (of M letters). For the model of permutations (written as words over $\{1,\dots,n\}$) and of words with geometric probabilities attached to the letters, we only provide the expected value. The reason is that the computations are too involved. A different and more effective approach would be most welcome.

2. Finite alphabet

We consider the model of an alphabet $\{1,\dots,M\}$ where each letter occurs with probability $\frac{1}{M}$, and different letters are independent from each other.

We note that $\mathbb{E}(\chi_{j,k})$ is just the probability that the box condition is satisfied between letters w_j and w_k.

This probability is not hard to compute: Let b be the larger and a be the smaller element of w_j and w_k. This introduces a factor 2 by symmetry. The letters $w_{j+1},\dots,w_{k-1}$ must be in the interval (a,b). Thus

$$\begin{aligned}\mathbb{E}(\mathsf{Box})&=2\sum_{1\le j<k\le n}\frac{1}{M^{k+1-j}}\sum_{1\le a<b\le M}(b-1-a)^{k-1-j}\\ &\doteq\sum_{1\le a<b\le M}\Big[\frac{2n}{M(M+a+1-b)}-\frac{2}{(M+a+1-b)^2}\Big]\\ &=\sum_{1\le k\le M}(M-k)\Big[\frac{2n}{M(M+1-k)}-\frac{2}{(M+1-k)^2}\Big]\\ &=\sum_{0\le k\le M-1}k\Big[\frac{2n}{M(k+1)}-\frac{2}{(k+1)^2}\Big]\\ &=\sum_{1\le k\le M}(k-1)\Big[\frac{2n}{Mk}-\frac{2}{k^2}\Big]\\ &=2n-2n\frac{H_M}{M}-2H_M+2H_M^{(2)}.\end{aligned}$$

We use here harmonic numbers (of arbitrary order)

$$H_M^{(d)} = \sum_{1\le k\le M} \frac{1}{k^d}.$$

Also, we have discarded exponentially small terms. For that, we use the notation $\doteq$, which means "equal, apart from exponentially small terms."

Now for the computation of the variance, we compute the second (factorial) moment: we have to compute

$$\mathbb{E}(\chi_{j,k}\cdot\chi_{l,m}),$$

where $1 \le j < k \le n$, $1 \le l < m \le n$, and $(j,k) \ne (l,m)$. Unfortunately, there are many cases to be distinguished, according to the pairs of indices. We distinguish 6 cases, and 6 other ones, which are symmetric, so that the cumulated results of the 6 cases (listed below) must be multiplied by 2.

Here are the 6 ranges of summation:

(1) $\{1 \le j < k < l < m \le n\}$,
(2) $\{1 \le j < l < m < k \le n\}$,
(3) $\{1 \le j < l < k < m \le n\}$,
(4) $\{1 \le j < k = l < m \le n\}$,
(5) $\{1 \le j < l < k = m \le n\}$,
(6) $\{1 \le j = l < k < m \le n\}$;

the 6 other ones are obtained by the replacements $j \leftrightarrow l$ and $k \leftrightarrow m$.

We summarize the contributions of these ranges below; the rewriting into harmonic numbers is nontrivial, and will be discussed later. We need more notation for that:

$$H_M^{(a,b)} := \sum_{1\le j<k\le M} \frac{1}{j^a k^b}, \quad H_M^{(a,b,c)} := \sum_{1\le j<k<l\le M} \frac{1}{j^a k^b l^c}.$$

In some rare cases, simplifications are possible:

$$H_M^{(1,1)} = \frac{1}{2}H_M^2 - \frac{1}{2}H_M^{(2)}.$$

The reader might like to consult [3] for reductions of such "Euler" sums.

- Range 1:

  With the abbreviations $f = M - b + 1 + a$, $g = M - d + 1 + c$:

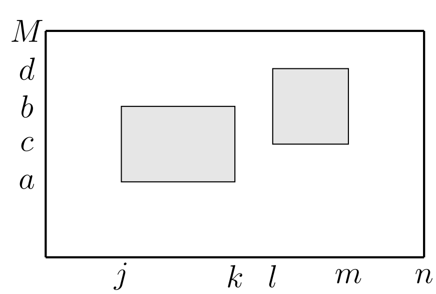

FIGURE 1. First range of summations.

$$
\begin{aligned}
R_1 &= 4\sum_{1\le j<k<l<m\le n}\frac{1}{M^{k+1-j+m+1-l}}\sum_{1\le a<b\le M}(b-1-a)^{k-1-j}\sum_{1\le c<d\le M}(d-1-c)^{m-1-l}\\
&\doteq \sum_{1\le a<b\le M}\sum_{1\le c<d\le M}\Big[\frac{2n^2}{M^2fg}-\frac{2n}{M^2}\frac{2Mf+fg+2Mg}{f^2g^2}+\frac{4(fg+f^2+g^2)}{f^3g^3}\Big]\\
&= n^2\Big[2-4\frac{H_M}{M}+2\frac{H_M^2}{M^2}\Big]\\
&- 2n\Big[4H_M+1-4H_M^{(2)}-\frac{2H_M}{M}-\frac{4H_M^2}{M}+\frac{H_M^2}{M^2}+\frac{4H_MH_M^{(2)}}{M}\Big]\\
&+ 4H_M^2+8MH_M^{(2)}-16H_MH_M^{(2)}-8MH_M^{(3)}+4(H_M^{(2)})^2+8H_MH_M^{(3)}.
\end{aligned}
$$

- Range 2:
  With the abbreviations $f=M-b+1+a$, $g=M-d+1+c$:

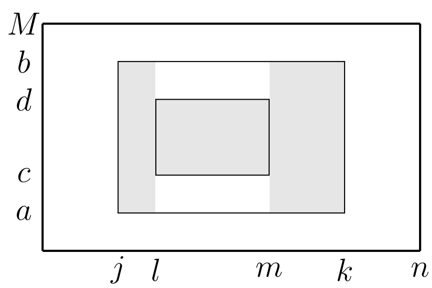

FIGURE 2. Second range of summations.

$$
\begin{aligned}
R_2 &= 4\sum_{1\le j<l<m<k\le n}\frac{1}{M^{k+1-j}}\sum_{1\le a<c<d<b\le M}(b-1-a)^{l-1-j+k-1-m}(d-1-c)^{m-1-l}\\
&\doteq \sum_{1\le a<c<d<b\le M}\Big[\frac{4n}{Mgf^2}-\frac{4}{g^2f^2}-\frac{8}{gf^3}\Big]\\
&= n\Big[4H_M+4\frac{H_M^{(2,1)}}{M}-8+8\frac{H_M}{M}-4H_M^{(2)}\Big]\\
&+ 2H_M^2-14H_M^{(2)}+12H_M+4H_M^{(2,1)}-2\big(H_M^{(2)}\big)^2+2H_M^{(4)}-8MH_M^{(2)}+8MH_M^{(3)}-8H_M^{(3,1)}.
\end{aligned}
$$

- Range 3:
  With the abbreviations $f=M-b+1+a$, $g=M-d+1+c$, $h=M-b+1+c$:

$$
\begin{aligned}
R_3 &= 4\sum_{1\le j<l<k<m\le n}\frac{1}{M^{m+1-j}}\sum_{1\le a<c<b<d\le M}(b-1-a)^{l-1-j}(b-1-c)^{k-1-l}(d-1-c)^{m-1-k}\\
&\doteq \sum_{1\le a<c<b<d\le M}\Big[4\frac{n}{fhgM}-4\frac{1}{fgh^2}-4\frac{1}{fg^2h}-4\frac{1}{f^2gh}\Big]
\end{aligned}
$$

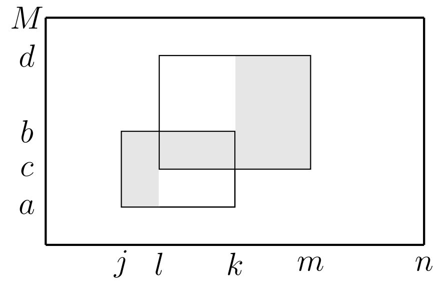

FIGURE 3. Third range of summations.

$$= \Big[2M - H_M + 2H_M^{(1,2)} - MH_M^{(2)} - H_M^{(2,1)}\Big]\frac{4n}{M}$$
$$- 4\Big[-2H_M^{(3,1)} - 2MH_M^{(3)} + H_M^{(2,1)} + 2MH_M^{(1,2)} + H_M^{(2,2)} + 2H_M^{(1,2,1)} + 6H_M^{(1,3)} + 2H_M\Big].$$

- Range 4:
  With the abbreviations $f = M - b + 1 + a$, $g = M - d + 1 + b$:

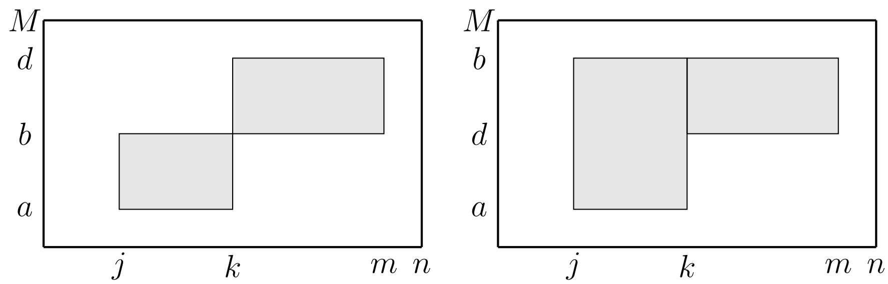

FIGURE 4. Fourth ranges of summations.

$$R_{4a} = 2\sum_{1\le j<k<m\le n}\frac{1}{M^{m+1-j}}\sum_{1\le a<b<d\le M}(b-1-a)^{k-1-j}(d-1-b)^{m-1-k}$$
$$\doteq \sum_{1\le a<b<d\le M}\Big[2\frac{n}{fgM} - 4\frac{1}{fg^2}\Big]$$
$$= \Big[\frac{2H_M}{M+1} - (M+2)H_M^{(2)} + 2M\Big]\frac{2n}{M}$$
$$- 4\Big[(M+2)H_M^{(1,2)} - (M+2)H_M^{(3)} + H_M + \frac{2}{(M+1)^2}H_M + H_M^{(2)} + \frac{1}{M+1}H_M^{(2)}\Big],$$

$$R_{4b} = 2\sum_{1\le j<k<m\le n}\frac{1}{M^{m+1-j}}\sum_{1\le a,d<b\le M}(b-1-a)^{k-1-j}(b-1-d)^{m-1-k}$$

$$\doteq \sum_{1\le a,d<b\le M}\Big[2\frac{n}{fgM}-4\frac{1}{fg^2}\Big]$$
$$= \Big[2M-H_M-H_M^2\Big]\frac{2n}{M}-2H_M^2+2H_M^{(2)}-4H_M+4H_MH_M^{(2)}.$$

- Range 5:
  With the abbreviations $f=M-b+1+a$, $g=M-b+1+c$:

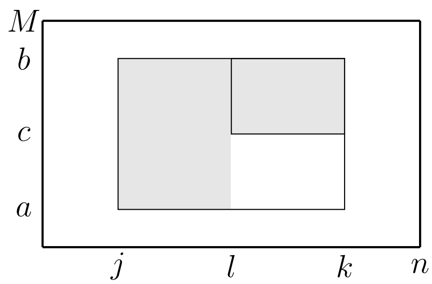

FIGURE 5. Fifth range of summations.

$$R_5 = 2\sum_{1\le j<l<k\le n}\frac{1}{M^{k+1-j}}\sum_{1\le a<c<b\le M}(b-1-a)^{l-1-j}(b-1-c)^{k-1-l}$$
$$\doteq \sum_{1\le a<c<b\le M}\Big[\frac{2n}{M}\frac{1}{fg}-\frac{2}{fg^2}-\frac{2}{f^2g}\Big]$$
$$= \Big[2M-H_M^2+H_M^{(2)}-2H_M\Big]\frac{n}{M}+2H_M^{(2,1)}-H_M^2+2H_M^{(1,2)}-2H_M+3H_M^{(2)}.$$

- Range 6:

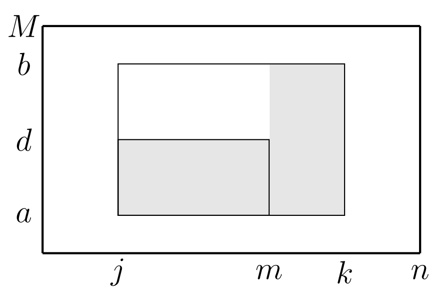

FIGURE 6. Sixth range of summations.

$$R_6 = R_5.$$

The second (factorial) moment is given by

$$2\Big(R_1+R_2+R_3+R_{4a}+R_{4b}+R_5+R_6\Big),$$

and the variance is

$$2\Big(R_1 + R_2 + R_3 + R_{4a} + R_{4b} + R_5 + R_6\Big) + E - E^2$$

with the expectated value $E = 2n - 2n\frac{H_M}{M} - 2H_M + 2H_M^{(2)}$.

We summarize the results.

Theorem 1. *The expectation and variance of the* box parameter *of random words of length n over an alphabet of M letters are (apart from exponentially small terms) given by*

$$\textsf{\textit{expectation}} = 2n - 2n\frac{H_M}{M} - 2H_M + 2H_M^{(2)},$$

$$\begin{aligned}\textsf{\textit{variance}} = n\Big(&- \frac{4H_M^2}{M^2} + \frac{2(M+5)H_M}{M(M+1)} - \frac{4(3M+1)H_M^{(2)}}{M} - \frac{8H_M H_M^{(2)}}{M} + \frac{16H_M^{(1,2)}}{M} + 22\Big)\\ &- \frac{2(9M^2+18M+17)}{(M+1)^2}H_M - \frac{2(9M+13)H_M^{(2)}}{M+1} + 8(3M+2)H_M^{(3)} - 8(3M+1)H_M^{(1,2)}\\ &- 8H_M^{(2,2)} + 4H_M^{(4)} + 8H_M^{(2,1)} - 48H_M^{(1,3)} - 16H_M^{(1,2,1)} - 16H_M H_M^{(2)} + 16H_M H_M^{(3)}.\end{aligned}$$

Now we discuss the simplifications needed here. The computations are quite long, so we decided to present just one representative instance and leave the others to the imagination of the reader. As Carsten Schneider[1] has kindly pointed out, he has software that can perform the calculations. However, we were ambitious enough to try it "by hand"; here is the hardest instance in full, appearing in range 3:

$$\begin{aligned}4\sum_{1\le j<l<k<m\le n}\frac{1}{M^{m+1-j}}&\sum_{1\le a<c<b<d\le M}(b-1-a)^{l-1-j}(b-1-c)^{k-1-l}(d-1-c)^{m-1-k}\\ &\doteq \sum_{1\le a<c<b<d\le M}\Big[-4\frac{1}{fgh^2} - 4\frac{1}{fg^2h} - 4\frac{1}{f^2gh} + 4\frac{n}{fghM}\Big],\end{aligned}$$

with the abbreviations for f, g, h as before. We show here how to compute

$$\mathsf{SUM} = \sum_{1\le a<c<b<d\le M}\frac{1}{fgh},$$

since it is the hardest term and thus the most instructive. We hope that the following computation also serves some pedagogic purpose.

$$\begin{aligned}\mathsf{SUM} &= \sum_{1\le a<c<b<d\le M}\frac{1}{(M-b+1+a)(M-b+1+c)(M-d+1+c)}\\ &= \sum_{1\le c<b\le M}\frac{(H_{M-b+c}-H_{M-b+1})(H_{M-b+c}-H_c)}{M-b+1+c}\end{aligned}$$

[1] His software Sigma struggled a bit with the present computation, but there is a new package EvaluateMultiSums based on Sigma that does it effortlessly; for details see [1] and references therein.

$$\begin{aligned}
&= \sum_{c=1}^{M-1} \sum_{b=1}^{M-c} \frac{(H_{M-b} - H_{M-b-c+1})(H_{M-b} - H_c)}{M-b+1} \\
&= \sum_{c=1}^{M-1} \sum_{b=c+1}^{M} \frac{(H_{b-1} - H_{b-c})(H_{b-1} - H_c)}{b} \\
&= \sum_{c=1}^{M-1} \sum_{b=c+1}^{M} \frac{H_{b-1}^2}{b} - \sum_{c=1}^{M-1} \sum_{b=c+1}^{M} \frac{H_{b-c}H_{b-1}}{b} - \sum_{c=1}^{M-1} \sum_{b=c+1}^{M} \frac{H_{b-1}H_c}{b} + \sum_{c=1}^{M-1} \sum_{b=c+1}^{M} \frac{H_{b-c}H_c}{b} \\
&= \sum_{b=2}^{M} \sum_{c=1}^{b-1} \frac{H_{b-1}^2}{b} - \sum_{b=2}^{M} \sum_{c=1}^{b-1} \frac{H_{b-c}H_{b-1}}{b} - \sum_{b=2}^{M} \sum_{c=1}^{b-1} \frac{H_{b-1}H_c}{b} + \sum_{b=2}^{M} \sum_{c=1}^{b-1} \frac{H_{b-c}H_c}{b} \\
&= \sum_{b=1}^{M} (b-1) \frac{H_{b-1}^2}{b} - \sum_{b=1}^{M} \frac{H_{b-1}}{b} \sum_{c=1}^{b-1} H_c - \sum_{b=1}^{M} \frac{H_{b-1}}{b} \sum_{c=1}^{b-1} H_c + \sum_{b=1}^{M} \frac{1}{b} \sum_{c=1}^{b-1} H_{b-c}H_c \\
&= \sum_{b=1}^{M} H_{b-1}^2 - \sum_{b=1}^{M} \frac{H_{b-1}^2}{b} - 2\sum_{b=1}^{M} \frac{H_{b-1}}{b} b(H_b - 1) + \sum_{b=1}^{M} \frac{1}{b} \sum_{c=1}^{b-1} H_{b-c}H_c \\
&= \sum_{b=1}^{M-1} H_b^2 - \sum_{b=1}^{M} \frac{H_{b-1}^2}{b} - 2\sum_{b=1}^{M} (H_b - \frac{1}{b})(H_b - 1) + \sum_{b=1}^{M} \frac{1}{b} [(b+1)(H_b^2 - H_b^{(2)}) - 2b(H_b - 1)] \\
&= \sum_{b=1}^{M} H_b^2 - H_M^2 - \sum_{b=1}^{M} \frac{H_{b-1}^2}{b} - 2\sum_{b=1}^{M} H_b^2 + 2\sum_{b=1}^{M} H_b \\
&+ 2\sum_{b=1}^{M} \frac{1}{b} H_b - 2\sum_{b=1}^{M} \frac{1}{b} + \sum_{b=1}^{M} [H_b^2 - H_b^{(2)} - 2(H_b - 1)] + \sum_{b=1}^{M} \frac{1}{b} [H_b^2 - H_b^{(2)}] \\
&= -\sum_{b=1}^{M} H_b^2 - H_M^2 - \sum_{b=1}^{M} \frac{(H_b - \frac{1}{b})^2}{b} + 2H_M + 2M(H_M - 1) + H_M^2 + H_M^{(2)} - 2H_M \\
&+ \sum_{b=1}^{M} [H_b^2 - H_b^{(2)}] - 2\sum_{b=1}^{M} H_b + 2M + \sum_{b=1}^{M} \frac{1}{b} [H_b^2 - H_b^{(2)}] \\
&= -\sum_{b=1}^{M} H_b^2 - H_M^2 - \sum_{b=1}^{M} \frac{H_b^2}{b} + 2\sum_{b=1}^{M} \frac{H_b}{b^2} - \sum_{b=1}^{M} \frac{1}{b^3} + 2H_M + 2M(H_M - 1) + H_M^2 \\
&+ H_M^{(2)} - 2H_M + \sum_{b=1}^{M} [H_b^2 - H_b^{(2)}] - 2M(H_M - 1) - 2H_M + 2M + \sum_{b=1}^{M} \frac{1}{b} [H_b^2 - H_b^{(2)}] \\
&= -[(M+1)H_M^2 - (2M+1)H_M + 2M] - H_M^2 + 2H_M^{(1,2)} + H_M^{(3)} + 2H_M + 2MH_M \\
&+ H_M^2 + H_M^{(2)} - 2H_M + 2\sum_{1 \le i < j \le b \le M} \frac{1}{ij} - 2MH_M - 2H_M + 2M - \sum_{b=1}^{M} \frac{H_b^{(2)}}{b} \\
&= -(M+1)H_M^2 + (2M+1)H_M - H_M^2 + 2H_M^{(1,2)} + H_M^{(3)} + 2MH_M + H_M^2 + H_M^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{1\le i<j\le M}\frac{M+1-j}{ij}-2MH_M-2H_M-H_M^{(2,1)}-H_M^{(3)}\\
&=-(M+1)H_M^2+2MH_M+H_M-H_M^2+2H_M^{(1,2)}+H_M^{(3)}+2MH_M+H_M^2+H_M^{(2)}\\
&+2(M+1)\sum_{1\le i<j\le M}\frac{1}{ij}-2\sum_{1\le i<j\le M}\frac{1}{i}-2MH_M-2H_M-H_M^{(2,1)}-H_M^{(3)}\\
&=-(M+1)H_M^2+2MH_M+H_M-H_M^2+2H_M^{(1,2)}+H_M^{(3)}+2MH_M+H_M^2+H_M^{(2)}\\
&+2(M+1)\sum_{1\le i<j\le M}\frac{1}{ij}-2\sum_{1\le i\le M}\frac{M-i}{i}-2MH_M-2H_M-H_M^{(2,1)}-H_M^{(3)}\\
&=-H_M+2H_M^{(1,2)}-MH_M^{(2)}+2M-H_M^{(2,1)}.
\end{aligned}
$$

3. Permutations

In the instance of permutations, we only provide the expected value. The computations for the variance were already quite formidable in the previous paper [2]; here, it would be much worse, and we leave this to a younger person, perhaps within a Ph.D. project. In all these problems it is noticeable that the expected value is not too difficult to compute, while the variance is orders of magnitudes more complicated.

It is not hard to write an expression for the expected value, where all permutations of n letters are equally likely:

$$E_n=2\sum_{1\le j<k\le n}\frac{1}{n^{\underline{k+1-j}}}\sum(b-1-a)^{\underline{k-1-j}};$$

the second sum runs over all $1\le a<b\le n$ such that $k-j\le b-a$. We use the notation $x^{\underline{m}}:=x(x-1)\dots(x-m+1)$ for the falling factorials.

The simplification is not difficult and best done by a computer:

$$
\begin{aligned}
E_n&=2\sum_{1\le j<k\le n}\Big[\frac{1}{k-j}-\frac{1}{k+1-j}\Big]\\
&=2\sum_{1\le j\le n}\Big[H_{n-j}-H_{n+1-j}+1\Big]\\
&=2\sum_{1\le j\le n-1}H_j-2\sum_{1\le j\le n}H_j+2n=2n-2H_n.
\end{aligned}
$$

4. Geometrically distributed words

Now we consider words where the letter k appears with (geometric) probability pq^{k-1}, with $p+q=1$, and the letters are independent from each other. Then we compute the average value of the box parameter. Again, the variance is an arduous task that we do not consider here. It is not out of reach, but requires a lot of dedication and concentration.

$$\begin{aligned}
E_n &= 2\sum_{1\le j<k\le n}\sum_{1\le a<b-1} p^2q^{a+b-2}q^{a(k-1-j)}(1-q^{b-1-a})^{k-1-j}\\
&= 2\sum_{1\le l\le n}(n-l)\sum_{1\le a<b-1} p^2q^{a+b-2}(q^a-q^{b-1})^{l-1}\\
&= 2\sum_{1\le l\le n}(n-l)\sum_{1\le a<b-1} p^2q^{a+b-2}\sum_{0\le s<l}\binom{l-1}{s}(-1)^s q^{a(l-1-s)+(b-1)s}\\
&= 2\sum_{1\le l\le n}(n-l)\sum_{1\le a<b-1} p^2q^{a-2}\sum_{0\le s<l}\binom{l-1}{s}(-1)^s q^{a(l-1-s)+b(s+1)-s}\\
&= 2\sum_{1\le l\le n}(n-l)\sum_{1\le a} p^2q^{a-2}\sum_{0\le s<l}\binom{l-1}{s}(-1)^s q^{a(l-1-s)+(a+2)(s+1)-s}\frac{1}{1-q^{s+1}}\\
&= 2\sum_{1\le l\le n}(n-l)\sum_{1\le a} p^2\sum_{0\le s<l}\binom{l-1}{s}(-1)^s q^{a(l+1)+s}\frac{1}{1-q^{s+1}}\\
&= 2\sum_{0\le s<l\le n}(n-l)p^2\binom{l-1}{s}(-1)^s q^{(l+1)+s}\frac{1}{1-q^{l+1}}\frac{1}{1-q^{s+1}}.
\end{aligned}$$

Unfortunately, this explicit expression is not very attractive. It is, however, instructive to perform the limit of this for $q\to 1$. It is known that one gets then the corresponding result for permutations. The reason is simple: In the limit, there are no duplicate elements, and each relative ordering of the elements is equally likely, which is just a permutation. Let us check this directly:

$$\begin{aligned}
\lim_{q\to 1}E_n &= 2\sum_{0\le s<l\le n}(n-l)\binom{l-1}{s}(-1)^s\frac{1}{l+1}\frac{1}{s+1}\\
&= 2\sum_{0\le s<l\le n}(n-l)\binom{l}{s+1}(-1)^s\frac{1}{l+1}\frac{1}{l}\\
&= 2\sum_{1\le l\le n}(n-l)\frac{1}{l+1}\frac{1}{l}\\
&= 2\sum_{1\le l\le n}\frac{n}{l}-2\sum_{1\le l\le n}\frac{n+1}{l+1}\\
&= 2nH_n-2\sum_{1\le l\le n}\frac{n+1}{l}-2+2(n+1)\\
&= 2nH_n-2nH_n-2H_n+2n=2n-2H_n,
\end{aligned}$$

as predicted.

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