A BIJECTION BETWEEN TERNARY TREES AND A SUBCLASS OF MOTZKIN PATHS

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Abstract. A bijection between ternary trees with n nodes and a subclass of Motzkin paths of length $3n$ is given. This bijection can then be generalized to t-ary trees.

1. Introduction

A recent question in the International Mathematics Competition proposed by Petrov and Vershik [4] counts the number of allowed paths from $(0, 0, 0)$ to (n, n, n) of a frog that makes steps of length one along the lattice

$$\Omega = \{(x, y, z) \in \mathbb{Z}^3 \mid 0 \le z \le y \le x \le y + 1\}$$

in exactly $3n$ moves.

Clearly there are n steps in each of the three possible directions, and we model each step as follows:

(x, y, z)	$(0, 0, 1)$	$(0, 1, 0)$	$(1, 0, 0)$
Step	$\diagdown$	$\diagup$	—

This along with the restriction $0 \le z \le y \le x \le y + 1$, gives rise to the subclass of Motzkin paths defined below.

Definition 1. *An* ***S-Motzkin path*** *is a Motzkin path with n of each type of step such that the following conditions hold*

- *The initial step must be* — *,*
- *between every two* — *there is exactly one* $\diagup$ *,*
- *the k-th occurring* $\diagdown$ *must occur after at least k pairs of* — *and* $\diagup$ *.*

The total number of such paths is $\frac{1}{2n+1}\binom{3n}{n}$ which is equal to the number of ternary trees with n nodes [1]. We first provide a mapping from S-Motzkin paths to ternary trees, and then provide the inverse mapping, thus showing that S-Motzkin paths are bijective to ternary trees as well as other combinatorial objects found in [2, 3, 5]. For completeness, an instructive example is given along with a table for $n = 3$.

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2. Bijection

2.1. **S-Motzkin paths to ternary trees.** We define $\varnothing$ to be the empty path. For an arbitrary S-Motzkin path $\mathcal{M}$, the canonical decomposition is

$$\Phi(\mathcal{M}) = (\mathcal{A},\, \mathcal{B},\, \mathcal{C})\,,$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ represent paths at the left, middle, and right subtrees respectively. Furthermore,

- $\mathcal{C}$ is the path from the penultimate return of the path to the last return, with the initial and last steps removed,
- $\mathcal{A}$ is the path from y to x (not including x), where x is the first $\_$ to the left of $\mathcal{C}$, and y the farthest away $\_$ from x such that the path from y to x is still a Motzkin path, and
- $\mathcal{B}$ is the path that remains after removing the path from the first return of the path from the right and the Motzkin path from y to x (including x) from the original path.

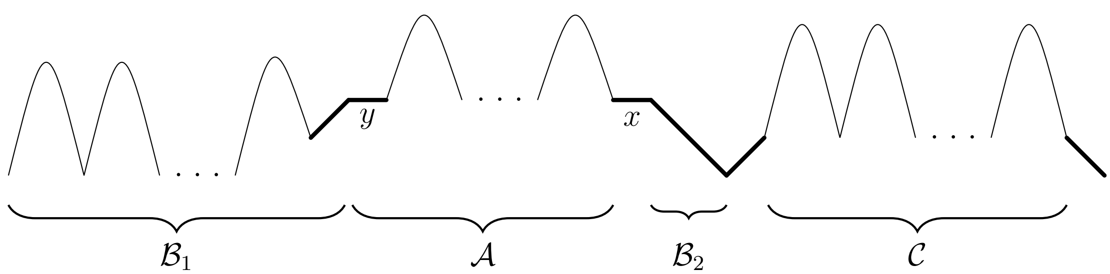


Figure 1. Canonical decomposition

This process is performed recursively and terminates at an empty path. Note that each application of Φ adds one node and removes one of each type of step. This proves inductively that an S-Motzkin path of length $3n$ maps to a ternary tree with n nodes.

2.2. **Ternary trees to S-Motzkin paths.** The inverse mapping is performed recursively bottom-up as follows. Each node of a ternary tree has three subtrees. Call the paths associated with the left, middle, and right subtrees $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively.

Starting at the end nodes, replace each node with

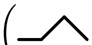
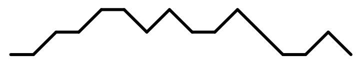
$$\mathcal{B}_1\,\mathcal{A}\,\_\,\mathcal{B}_2 \diagup \mathcal{C} \diagdown\,,$$

where $\mathcal{B}_1$ is the subpath of $\mathcal{B}$ that starts at $(0,0)$ and extends to and includes the first occurring $\diagup$ from the right. The path $\mathcal{B}_2$ is what remains of $\mathcal{B}$ after removing $\mathcal{B}_1$.

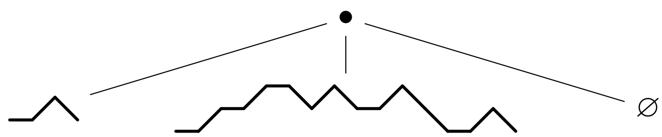
This process is continued recursively on each set of end nodes and terminates at the root to produce an S-Motzkin path. Note that for each node three steps are added, and thus a ternary tree with n nodes produces an S-Motzkin path of length $3n$.

2.3. **Example.** As an example, we map the following S-Motzkin path into a ternary tree. Since the steps are reversible, the inverse mapping can be seen by reading the example in reverse. Let $\mathcal{M}$ be

The canonical decomposition of $\mathcal{M}$ is

$\Phi(\mathcal{M}) = \Big($  ,  , $\varnothing\Big).$

Hence



Continuing recursively:

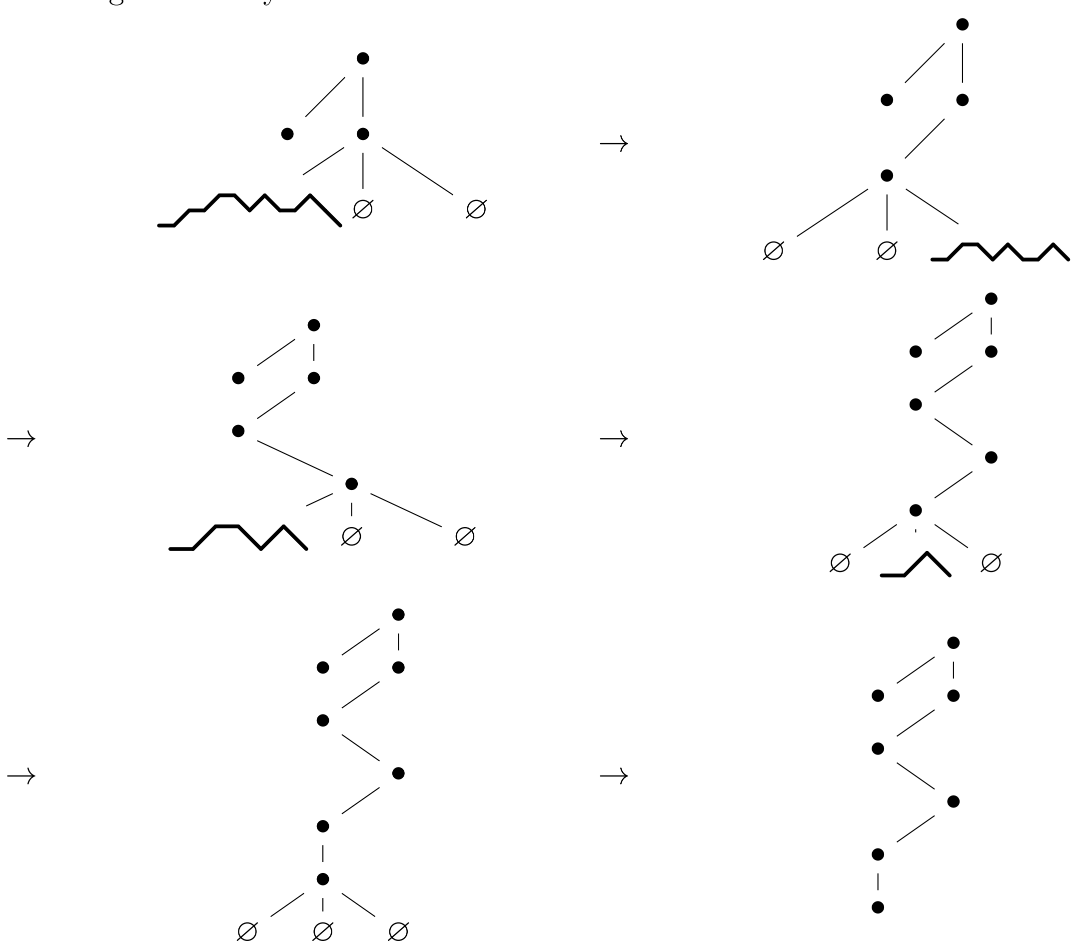
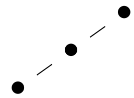
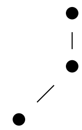
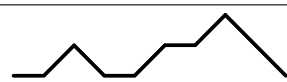
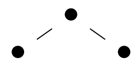
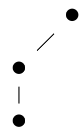
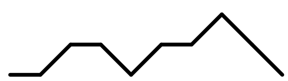
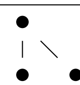
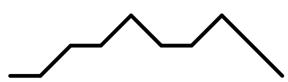
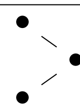
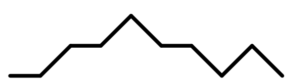
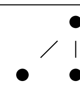
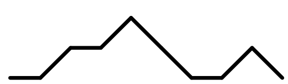
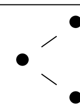
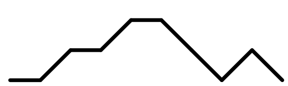
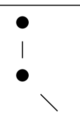
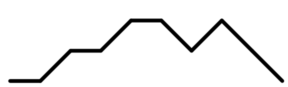
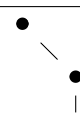
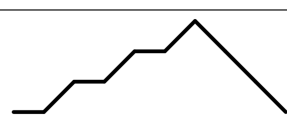
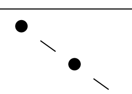


TABLE 1. Bijection for $n = 3$

S-Motzkin path	Ternary tree
	
	
	
	
	
	
	
	
	
	
	
	

3. Generalization

This bijection can be generalized to be between t-ary trees and the subclass of Motzkin paths with $(t-2)n$ $\_$ steps and n of each of the other steps such that

- The initial $t-2$ steps must be of the form $\_$,
- between every two $\diagup$ there are exactly $t-2$ steps of the form $\_$,
- the k-th occurring $\diagdown$ must occur after at least k occurrences of $t-2$ steps of the form $\_$ and one step of the form $\diagup$.

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