# CONTINUED FRACTION EXPANSIONS RELATED TO GÖLLNITZ' LITTLE PARTITION THEOREM

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Abstract. Previous lists of continued fraction expansions related to series of the Rogers-Ramanujan type are augmented by some instances derived from Göllnitz' little partition theorem.

## 1. Introduction

One of the present authors has been trying to collect as many beautiful continued fraction expansions as possible for quotients of series which resemble the celebrated Rogers-Ramanujan identities [3, 4, 2].

However, a few nice examples have been overlooked. The present note addresses them; they are motivated by the *little theorem of Göllnitz* [1]:

$$\sum_{n\ge0}\frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n}=(-q^2;q^4)_\infty(-q^3;q^4)_\infty(-q^4;q^4)_\infty.$$

$$\sum_{n\ge0}\frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n}=(-q;q^4)_\infty(-q^2;q^4)_\infty(-q^4;q^4)_\infty.$$

We use standard notation: $(x;q)_n=\prod_{i=1}^{n}(1-xq^{i-1})$, where $n$ is either a non-negative integer or $\infty$.

Set

$$F(z,w)=\sum_{n\ge0}z^n\frac{q^{n^2}(-qw;q^2)_n}{(q^2;q^2)_n},$$

$$G(z,w)=\sum_{n\ge0}z^n\frac{q^{n^2}(-q^{-1}w;q^2)_n}{(q^2;q^2)_n},$$

then $F(q,1)$ and $G(q,1)$ are the series in the little theorem of Göllnitz. We found the continued fraction expansion of $zF(z,w)/G(z,w)$. Likewise, we found some variations by replacing $(q^2;q^2)_n$ in the denominators by $(q;q)_{2n(+1)}$. Also we investigated instances where $q^{n^2}$ isn't present. Typically, such series (without the factor $q^{n^2}$) are easier, and it is not unlikely that our continued fraction results for them are known and/or derivable from known formulæ. However, they can easily be derived in the style of the other expansions in our note, so we just state them for completeness.

Our method is as follows: We set $s_0=F$, $s_{-1}=G$, and

$$s_{k+1}=\frac{s_{k-1}-a_ks_k}{z};$$

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the numbers $a_k$ are uniquely defined to make all the $s_k$'s power series. They are guessed. After that, the series $s_k$ are also guessed. Once all these quantities are known, a proof by induction is just routine. We will show this on one example, and for the other instances, just present the $a_k$ and $s_k$. Then:

$$\frac{zF}{G} = \frac{zs_0}{a_0s_0 + zs_1} = \cfrac{z}{a_0 + \cfrac{zs_1}{a_1s_1 + zs_2}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{zs_2}{a_2s_2 + zs_3}}} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}}.$$

## 2. Results

**Theorem 1.** *Set*

$$F = \sum_{n\ge0} z^n \frac{q^{n^2}(-qw;q^2)_n}{(q^2;q^2)_n}, \qquad G = \sum_{n\ge0} z^n \frac{q^{n^2}(-q^{-1}w;q^2)_n}{(q^2;q^2)_n}.$$

*Then*

$$\frac{zF}{G} = \cfrac{z}{a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\cdots}}}},$$

*with*

$$a_{2k} = \frac{q^{k^2-2k}w^k}{(-qw;q^2)_k},$$
$$a_{2k+1} = \frac{q^{-k^2-2k}(-qw;q^2)_k}{w^{k+1}}.$$

*The auxiliary series are*

$$s_{2k} = q^{k^2} \sum_{n\ge0} z^n \frac{q^{n^2+2nk}(-qw;q^2)_{n+k}}{(q^2;q^2)_n},$$
$$s_{2k+1} = q^{2k^2+2k} \sum_{n\ge0} z^n \frac{q^{n^2+2n(k+1)}(-q^{1+2k}w;q^2)_n w^{k+1}}{(q^2;q^2)_n}.$$

*Proof.* We consider for $n \ge 1$

$$\begin{aligned}
[z^{n-1}]\frac{s_{2k-1} - a_{2k}s_{2k}}{z} &= [z^n]s_{2k-1} - a_{2k}[z^n]s_{2k} \\
&= q^{2(k-1)^2+2(k-1)} \frac{q^{n^2+2nk}(-q^{-1+2k}w;q^2)_n w^k}{(q^2;q^2)_n} \\
&\quad - q^{2k^2-2k}w^k \frac{q^{n^2+2nk}(-q^{1+2k}w;q^2)_n}{(q^2;q^2)_n} \\
&= \frac{q^{2k^2-2k+n^2+2nk}(-q^{1+2k}w;q^2)_{n-1}w^k}{(q^2;q^2)_n}\Big[(1+q^{-1+2k}w) - (1+q^{-1+2k+2n}w)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^{2k^2-1+n^2+2nk}(-q^{1+2k}w;q^2)_{n-1}w^{k+1}}{(q^2;q^2)_n}(1-q^{2n})\\
&= \frac{q^{2k^2+2k+(n-1)^2+2(n-1)(k+1)}(-q^{1+2k}w;q^2)_{n-1}w^{k+1}}{(q^2;q^2)_{n-1}}\\
&= [z^{n-1}]s_{2k+1}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
[z^{n-1}]\frac{s_{2k}-a_{2k+1}s_{2k+1}}{z} &= [z^n]s_{2k}-a_{2k+1}[z^n]s_{2k+1}\\
&= \frac{q^{n^2+2nk+k^2}(-qw;q^2)_{n+k}}{(q^2;q^2)_n} - \frac{q^{n^2+2n(k+1)+k^2}(-qw;q^2)_{n+k}}{(q^2;q^2)_n}\\
&= \frac{q^{n^2+2nk+k^2}(-qw;q^2)_{n+k}}{(q^2;q^2)_{n-1}}\\
&= [z^{n-1}]s_{2k+2}.
\end{aligned}
$$

These computations also show that the constant term in $s_{k-1}-a_ks_k$ vanishes, thus our (model) proof is finished. $\square$

The continued fractions often look more attractive when they are transformed:

$$
\cfrac{z}{a_0+\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{\dots}}}} = \cfrac{z/a_0}{1+\cfrac{z/a_0a_1}{1+\cfrac{z/a_1a_2}{1+\cfrac{z/a_2a_3}{\dots}}}}
$$

We show this only for the result in the previous theorem, which takes the form

$$
\frac{zF}{G} = \cfrac{z}{1+\cfrac{zw}{1+\cfrac{z(1+qw)q}{1+\cfrac{zwq^4}{1+\cfrac{z(1+q^3w)q^3}{\dots}}}}}.
$$

**Theorem 2.** *Set*

$$
F=\sum_{n\ge0} z^n\frac{q^{n^2}(-qw;q^2)_n}{(q;q)_{2n}}, \qquad G=\sum_{n\ge0} z^n\frac{q^{n^2}(-q^{-1}w;q^2)_n}{(q;q)_{2n}}.
$$

*Then*

$$
\begin{aligned}
a_{2k} &= \frac{q^{k^2-2k}(-q^2/w;q^2)_{k-1}(1-q^{4k-1})w^k}{(-qw;q^2)_k},\ k\ge1, \quad a_0=1,\\
a_{2k+1} &= \frac{q^{-k^2-2k}(-qw;q^2)_k(1-q^{4k+1})}{(-q^2/w;q^2)_kw^{k+1}}
\end{aligned}
$$

*and*

$$s_{2k}=q^{k^2}\sum_{n\ge0}z^n\frac{q^{n^2+2nk}(-qw;q^2)_k(-q;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}(-q;q^2)_k},$$
$$s_{2k+1}=q^{2k^2+2k}\sum_{n\ge0}z^n\frac{q^{n^2+2n(k+1)}(-q;q^2)_{n+k}(-q^2/w;q^2)_kw^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}(-q;q^2)_k}.$$

**Theorem 3.** *Set*

$$F=\sum_{n\ge0}z^n\frac{(-qw;q^2)_n}{(q;q)_{2n}},\qquad G=\sum_{n\ge0}z^n\frac{(-q^{-1}w;q^2)_n}{(q;q)_{2n}}.$$

*Then*

$$a_{2k}=\frac{q^{-k^2-k}(-q^2/w;q^2)_{k-1}(1-q^{4k-1})w^k}{(-qw;q^2)_k},\ k\ge1,\quad a_0=1,$$
$$a_{2k+1}=\frac{q^{k^2-k+1}(-qw;q^2)_k(1-q^{4k+1})}{(-q^2/w;q^2)_kw^{k+1}},$$

*and*

$$s_{2k}=q^{2k^2-k}\sum_{n\ge0}z^n\frac{(-qw;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}},$$
$$s_{2k+1}=q^{k^2-1}\sum_{n\ge0}z^n\frac{(-q^{1+2k}w;q^2)_n(-q^2/w;q^2)_kw^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}}.$$

**Theorem 4.** *Set*

$$F=\sum_{n\ge0}z^n\frac{q^{n^2}(-qw;q^2)_n}{(q;q)_{2n+1}},\qquad G=\sum_{n\ge0}z^n\frac{q^{n^2}(-q^{-1}w;q^2)_n}{(q;q)_{2n}}.$$

*Then*

$$a_{2k}=\frac{q^{k^2-2k}(-q^2/w;q^2)_k(1-q^{4k+1})w^k}{(-qw;q^2)_k},$$
$$a_{2k+1}=\frac{q^{-k^2-2k}(-qw;q^2)_k(1-q^{4k+3})}{(-q^2/w;q^2)_{k+1}w^{k+1}}$$

*and*

$$s_{2k}=q^{k^2}\sum_{n\ge0}z^n\frac{q^{n^2+2nk}(-qw;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}(1-q^{4k+2n+1})},$$
$$s_{2k+1}=q^{2k^2+2k}\sum_{n\ge0}z^n\frac{q^{n^2+2n(k+1)}(-q^{1+2k};q^2)_n(-q^2/w;q^2)_{k+1}w^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}(1-q^{4k+2n+3})}.$$

**Theorem 5.** *Set*

$$F=\sum_{n\ge0}z^n\frac{(-qw;q^2)_n}{(q;q)_{2n+1}},\qquad G=\sum_{n\ge0}z^n\frac{(-q^{-1}w;q^2)_n}{(q;q)_{2n}}.$$

*Then*

$$a_{2k} = \frac{q^{-k^2-3k}(-q^2/w;q^2)_k(1-q^{4k+1})w^k}{(-qw;q^2)_k},$$
$$a_{2k+1} = \frac{q^{k^2+k+1}(-qw;q^2)_k(1-q^{4k+3})}{(-q^2/w;q^2)_{k+1}w^{k+1}}$$

*and*

$$s_{2k} = q^{2k^2+k}\sum_{n\geq 0} z^n \frac{(-qw;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}(1-q^{4k+2n+1})},$$
$$s_{2k+1} = q^{k^2-1}\sum_{n\geq 0} z^n \frac{(-q^{1+2k};q^2)_n(-q^2/w;q^2)_{k+1}w^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}(1-q^{4k+2n+3})}.$$

**Theorem 6.** *Set*

$$F = \sum_{n\geq 0} z^n \frac{q^{n^2}(-qw;q^2)_n}{(q;q)_{2n+1}}, \qquad G = \sum_{n\geq 0} z^n \frac{q^{n^2}(-q^{-1}w;q^2)_n}{(q;q)_{2n+1}}.$$

*Then*

$$a_{2k} = \frac{q^{k^2-2k}(-q^4/w;q^2)_{k-1}(1-q^{4k+1})w^k}{(-qw;q^2)_k}, \ k\geq 1, \quad a_0 = 1,$$
$$a_{2k+1} = \frac{q^{-k^2-2k}(-qw;q^2)_k(1-q^{4k+3})}{(-q^4/w;q^2)_k w^{k+1}}$$

*and*

$$s_{2k} = q^{k^2}\sum_{n\geq 0} z^n \frac{q^{n^2+2nk}(-qw;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}(1-q^{4k+2n+1})},$$
$$s_{2k+1} = q^{2k^2+2k}\sum_{n\geq 0} z^n \frac{q^{n^2+2n(k+1)}(-q^{1+2k};q^2)_n(-q^4/w;q^2)_k w^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}(1-q^{4k+2n+3})}.$$

**Theorem 7.** *Set*

$$F = \sum_{n\geq 0} z^n \frac{(-qw;q^2)_n}{(q;q)_{2n+1}}, \quad G = \sum_{n\geq 0} z^n \frac{(-q^{-1}w;q^2)_n}{(q;q)_{2n+1}}.$$

*Then*

$$a_{2k} = \frac{q^{-k^2-3k}(-q^4/w;q^2)_{k-1}(1-q^{4k+1})w^k}{(-qw;q^2)_k}, \ k\geq 1, \quad a_0 = 1$$
$$a_{2k+1} = \frac{q^{k^2+k+1}(-qw;q^2)_k(1-q^{4k+3})}{(-q^4/w;q^2)_k w^{k+1}}$$

*and*

$$s_{2k} = q^{k^2}\sum_{n\geq 0} z^n \frac{(-qw;q^2)_{n+k}}{(q^2;q^2)_n(q;q^2)_{2k+n}(1-q^{4k+2n+1})},$$
$$s_{2k+1} = q^{k^2-1}\sum_{n\geq 0} z^n \frac{(-q^{1+2k};q^2)_n(-q^4/w;q^2)_k w^{k+1}}{(q^2;q^2)_n(q;q^2)_{2k+n+1}(1-q^{4k+2n+3})}.$$

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