SECANT AND COSECANT SUMS AND BERNOULLI-NÖRLUND POLYNOMIALS

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Abstract. We give explicit formulæ for sums of even powers of secant and cosecant values in terms of Bernoulli numbers and central factorial numbers.

1. Introduction

We derive explicit formulæ for the *secant sum*

$$S_{2m}(N) := \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\cos^{2m} \frac{k\pi}{N}}$$

and the *cosecant sum*

$$C_{2m}(N) := \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \frac{1}{\sin^{2m} \frac{k\pi}{N}}.$$

This research is inspired by the paper [2], where such formulæ were given for $m \leq 6$. Our approach, which uses contour integrals and residues, produces such formulæ quite effortlessly for any m. The main contribution of the present paper is the identification of the occurring coefficients as "classical" combinatorial quantities such as *central factorial numbers* and *Bernoulli numbers*.

2. Contour integrals and residues

We consider the *secant sum* first and start with the contour integral

$$\frac{1}{2\pi i} \oint_{R_T} \frac{1}{\cos^{2m} \pi z} \pi N \cot(\pi N z)\, dz, \tag{1}$$

where R_T is the rectangle with corners $-\frac{1}{2N} \pm iT$, $1 - \frac{1}{2N} \pm iT$. By periodicity of the integrand, the integrals along the vertical lines cancel. Furthermore, the integrals along the horizontal lines tend to 0 when $T \to \infty$, since cot remains bounded and cos tends to infinity exponentially.

† This author is supported by the grant S9605-N12 of the Austrian Science Fund FWF.
* This author is supported by the grant NRF 2053748 of the South African National Research Foundation.

Thus we have

$$0 = \frac{1}{2\pi i}\oint_{R_T} \frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z)\,dz$$
$$= 2\sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{\cos^{2m}\frac{k\pi}{N}} + 1 + \operatorname*{Res}_{z=\frac12}\frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z)$$

by the residue theorem. From this we derive

$$S_{2m}(N) = \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{\cos^{2m}\frac{k\pi}{N}} = -\frac12 - \frac12\operatorname*{Res}_{z=\frac12}\frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z). \tag{2}$$

In [4] the Bernoulli-Nörlund polynomials are introduced by the relation

$$\frac{\omega_1\cdots\omega_k t^k e^{xt}}{(e^{\omega_1 t}-1)\cdots(e^{\omega_k t}-1)} = \sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(k)}(x;\omega_1,\dots,\omega_k). \tag{3}$$

We specialise $\omega_1 = \cdots = \omega_k = 2i$, $x = ki$, and $t = \pi z$ to obtain

$$\left(\frac{\pi z}{\sin\pi z}\right)^k = \sum_{n=0}^{\infty}\frac{(\pi z)^n}{n!}B_n^{(k)}(ki;2i,\dots,2i).$$

Writing $P_n^{(k)} = i^n B_n^{(k)}(ki;2i,\dots,2i)$ and observing that $P_{2n+1}^{(k)} = 0$ we have

$$\frac{1}{\sin^k\pi z} = \sum_{n=0}^{\infty}\frac{(\pi z)^{2n-k}}{(2n)!}(-1)^n P_{2n}^{(k)}. \tag{4}$$

We have

$$\operatorname*{Res}_{z=\frac12}\frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z) = \operatorname*{Res}_{z=0}\frac{1}{\sin^{2m}\pi z}\pi N\cot(\pi N z + \frac{N}{2}\pi).$$

Notice that

$$\cot(\pi N z + \frac{N}{2}\pi) = \begin{cases}\cot(\pi N z) & \text{if } N \text{ is even},\\ -\tan(\pi N z) & \text{if } N \text{ is odd}.\end{cases}$$

Thus it is natural to distinguish two cases according to the parity of N.

From [3] we have

$$\pi\cot\pi z = \sum_{n=0}^{\infty}\frac{\pi^{2n}z^{2n-1}}{(2n)!}(-1)^n 4^n B_{2n}, \tag{5}$$
$$\pi\tan\pi z = \sum_{n=1}^{\infty}\frac{\pi^{2n}z^{2n-1}}{(2n)!}(-1)^{n-1}4^n(4^n-1)B_{2n}.$$

Then for even N we have

$$\begin{aligned}
&\operatorname*{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z)\\
&\qquad = [z^{-1}]\sum_{\ell=0}^{\infty}\frac{(\pi z)^{2\ell-2m}}{(2\ell)!}(-1)^{\ell}P_{2\ell}^{(2m)}\pi N\sum_{n=0}^{\infty}\frac{(N\pi z)^{2n-1}}{(2n)!}(-1)^n 4^n B_{2n}\\
&\qquad = \frac{(-1)^m}{(2m)!}\sum_{n=0}^{m}\binom{2m}{2n}P_{2(m-n)}^{(2m)}B_{2n}(2N)^{2n}
\end{aligned}$$

and for odd N

$$\begin{aligned}
&\operatorname*{Res}_{z=\frac{1}{2}} \frac{1}{\cos^{2m}\pi z}\pi N\cot(\pi N z)\\
&\qquad = -[z^{-1}]\sum_{\ell=0}^{\infty}\frac{(\pi z)^{2\ell-2m}}{(2\ell)!}(-1)^{\ell}P_{2\ell}^{(2m)}\pi N\sum_{n=1}^{\infty}\frac{(\pi N z)^{2n-1}}{(2n)!}(-1)^{n-1}4^n(4^n-1)B_{2n}\\
&\qquad = \frac{(-1)^m}{(2m)!}\sum_{n=1}^{m}\binom{2m}{2n}P_{2(m-n)}^{(2m)}B_{2n}(4^n-1)(2N)^{2n}.
\end{aligned}$$

Summing up, we have for even N

$$S_{2m}(N) = -\frac{1}{2} + \frac{(-1)^{m-1}}{2(2m)!}\sum_{n=0}^{m}\binom{2m}{2n}P_{2(m-n)}^{(2m)}B_{2n}(2N)^{2n} \tag{6}$$

and for odd N

$$S_{2m}(N) = -\frac{1}{2} + \frac{(-1)^{m-1}}{2(2m)!}\sum_{n=0}^{m}\binom{2m}{2n}P_{2(m-n)}^{(2m)}B_{2n}(4^n-1)(2N)^{2n}. \tag{7}$$

Equation (2) gives us for even N:

$$\begin{aligned}
m=1:&\quad \tfrac{1}{6}N^2-\tfrac{2}{3}\\
m=2:&\quad \tfrac{1}{90}N^4+\tfrac{1}{9}N^2-\tfrac{28}{45}\\
m=3:&\quad \tfrac{1}{945}N^6+\tfrac{1}{90}N^4+\tfrac{4}{45}N^2-\tfrac{568}{945}\\
m=4:&\quad \tfrac{1}{9450}N^8+\tfrac{4}{2835}N^6+\tfrac{7}{675}N^4+\tfrac{8}{105}N^2-\tfrac{8336}{14175}\\
m=5:&\quad \tfrac{1}{93555}N^{10}+\tfrac{1}{5670}N^8+\tfrac{13}{8505}N^6+\tfrac{82}{8505}N^4+\tfrac{64}{945}N^2-\tfrac{54176}{93555}\\
m=6:&\quad \tfrac{691}{638512875}N^{12}+\tfrac{2}{93555}N^{10}+\tfrac{31}{141750}N^8+\tfrac{278}{178605}N^6+\tfrac{1916}{212625}N^4+\tfrac{128}{2079}N^2-\tfrac{365470016}{638512875}\\
m=7:&\quad \tfrac{2}{18243225}N^{14}+\tfrac{691}{273648375}N^{12}+\tfrac{2}{66825}N^{10}+\tfrac{311}{1275750}N^8+\tfrac{592}{382725}N^6+\tfrac{944}{111375}N^4\\
&\qquad +\tfrac{512}{9009}N^2-\tfrac{155194496}{273648375}\\
m=8:&\quad \tfrac{3617}{325641566250}N^{16}+\tfrac{16}{54729675}N^{14}+\tfrac{113324}{28733079375}N^{12}+\tfrac{1072}{29469825}N^{10}+\tfrac{2473}{9568125}N^8\\
&\qquad +\tfrac{134432}{88409475}N^6+\tfrac{8533792}{1064188125}N^4+\tfrac{1024}{19305}N^2-\tfrac{274946646272}{488462349375}
\end{aligned}$$

Equation (2) gives us for odd N:

$m=1:$ $\frac{1}{2}N^2-\frac{1}{2}$

$m=2:$ $\frac{1}{6}N^4+\frac{1}{3}N^2-\frac{1}{2}$

$m=3:$ $\frac{1}{15}N^6+\frac{1}{6}N^4+\frac{4}{15}N^2-\frac{1}{2}$

$m=4:$ $\frac{17}{630}N^8+\frac{4}{45}N^6+\frac{7}{45}N^4+\frac{8}{35}N^2-\frac{1}{2}$

$m=5:$ $\frac{31}{2835}N^{10}+\frac{17}{378}N^8+\frac{13}{135}N^6+\frac{82}{567}N^4+\frac{64}{315}N^2-\frac{1}{2}$

$m=6:$ $\frac{691}{155925}N^{12}+\frac{62}{2835}N^{10}+\frac{527}{9450}N^8+\frac{278}{2835}N^6+\frac{1916}{14175}N^4+\frac{128}{693}N^2-\frac{1}{2}$

$m=7:$ $\frac{10922}{6081075}N^{14}+\frac{691}{66825}N^{12}+\frac{62}{2025}N^{10}+\frac{5287}{85050}N^8+\frac{592}{6075}N^6+\frac{944}{7425}N^4+\frac{512}{3003}N^2-\frac{1}{2}$

$m=8:$ $\frac{929569}{1277025750}N^{16}+\frac{87376}{18243225}N^{14}+\frac{113324}{7016625}N^{12}+\frac{33232}{893025}N^{10}+\frac{42041}{637875}N^8$
$+\frac{134432}{1403325}N^6+\frac{8533792}{70945875}N^4+\frac{1024}{6435}N^2-\frac{1}{2}$

For the *cosecant* sum, we start with the contour integral

$$\frac{1}{2\pi i}\oint_{R_T}\frac{1}{\sin^{2m}\pi z}\pi N\cot(\pi N z)dz, \tag{8}$$

which is again zero and, by summing residues, leads to the equation

$$0=\sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{\sin^{2m}\frac{k\pi}{N}}+\frac{1}{2}\operatorname*{Res}_{z=0}\frac{1}{\sin^{2m}\pi z}\pi N\cot(\pi N z)+\frac{1+(-1)^N}{4}.$$

We observe that

$$\sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{\sin^{2m}\frac{k\pi}{N}}+\frac{1+(-1)^N}{4}$$

equals the residue that we already calculated for $S_{2m}(N)$ and N even. Thus we have

$$C_{2m}(N)=\sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1}{\sin^{2m}\frac{k\pi}{N}}=\frac{(-1)^m}{(2m)!}\sum_{n=0}^{m}\binom{2m}{2n}P_{2(m-n)}^{(2m)}B_{2n}(2N)^{2n}-\frac{1+(-1)^N}{4}. \tag{9}$$

3. Computing $P_{2n}^{(2m)}$

In this section we want to have a closer look at the Laurent series expansion of $\sin^{-2m}\pi z$. Our approach is somewhat similar to the one used in [1].

We start with the expansion (5). Differentiating yields

$$\frac{1}{\sin^2\pi z}=\sum_{n=0}^{\infty}\frac{(\pi z)^{2n-2}}{(2n)!}(2n-1)(-1)^{n-1}4^nB_{2n}.$$

This gives

$$P_{2n}^{(2)}=-(2n-1)4^nB_{2n}. \tag{10}$$

Differentiating $\sin^{-2m}\pi z$ twice yields

$$\frac{d^2}{dz^2}\frac{1}{\sin^{2m}\pi z} = 2m(2m+1)\pi^2\frac{1}{\sin^{2m+2}\pi z} - 4m^2\pi^2\frac{1}{\sin^{2m}\pi z}. \tag{11}$$

We now write

$$\frac{1}{\sin^{2m}\pi z} = H_{2m}(z) + R_{2m}(z) = \frac{1}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(2\ell-1)!b_{2\ell}^{(2m)}4^{m-\ell}}{(\pi z)^{2\ell}} + R_{2m}(z), \tag{12}$$

where H_{2m} is the principal part around $z=0$ and R_{2m} denotes the regular part. Since differentiation preserves principal and regular parts, (11) gives

$$H_{2m}''(z) = \pi^2 2m(2m+1)H_{2m+2}(z) - 4m^2\pi^2 H_{2m}(z), \tag{13}$$

which gives the recursion (setting $b_0^{(2m)} = b_{2m+2}^{(2m)} = 0$ and $b_2^{(2)} = 1$)

$$b_{2\ell}^{(2m+2)} = m^2 b_{2\ell}^{(2m)} + b_{2\ell-2}^{(2m)} \text{ for } 1 \le \ell \le m+1. \tag{14}$$

This recursion shows that the numbers $b_{2\ell}^{(2m)}$ are given by

$$\sum_{\ell=0}^{m} b_{2\ell}^{(2m)}x^{2\ell} = \prod_{k=0}^{m-1}(x^2+k^2). \tag{15}$$

Thus they are closely related to the *central factorial* numbers $t(n,k)$ studied in [5, p. 213]:

$$x\prod_{k=1}^{m-1}(x^2-k^2) = \sum_{k=0}^{2m} t(2m,2k+1)x^{2k+1}$$

and a similar expression for odd first argument. This gives $b_{2\ell}^{(2m)} = (-1)^{\ell+m}t(2m,2\ell)$. We notice that the polynomials in (15) appear *mutatis mutandis* in [2] as differential operators. These operators are used to model the recursion (13).

In Table 1 we computed the values $b_{2k}^{(2m)}$ for small values of m.

$b_k^{(m)}$	$k=2$	4	6	8	10	12	14	16	18
$m=2$	1								
4	1	1							
6	4	5	1						
8	36	49	14	1					
10	576	820	273	30	1				
12	14400	21076	7645	1023	55	1			
14	518400	773136	296296	44473	3003	91	1		
16	25401600	38402064	15291640	2475473	191620	7462	140	1	
18	1625702400	2483133696	1017067024	173721912	14739153	669188	16422	204	1

TABLE 1. Table of $b_k^{(m)}$ for small values of m (compare with [5, Table 6.1, p. 217])

We now consider the Mittag-Leffler expansion

$$\frac{1}{\sin^{2m}\pi z}=\sum_{n\in\mathbb{Z}}H_{2m}(z+n)=H_{2m}(z)+\sum_{n=1}^{\infty}\big(H_{2m}(z+n)+H_{2m}(z-n)\big). \tag{16}$$

Expanding the last sum into a power series and using (12) yields

$$\frac{1}{\sin^{2m}\pi z}=H_{2m}(z)+\frac{4^m}{(2m-1)!}\sum_{k=0}^{\infty}\frac{(\pi z)^{2k}}{(2k)!}4^k(-1)^k\sum_{\ell=1}^{m}(-1)^{\ell-1}\frac{1}{2\ell+2k}b_{2\ell}^{(2m)}B_{2\ell+2k},$$

where we have used $\zeta(2k)=(-1)^{k-1}\frac{2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}$. This gives

$$P_{2k}^{(2m)}=\begin{cases}2m\binom{2k}{2m}4^k\sum_{\ell=0}^{m-1}(-1)^{\ell-1}\frac{1}{2k-2\ell}b_{2m-2\ell}^{(2m)}B_{2k-2\ell} & \text{for } k\geq m,\\ (-1)^k4^kb_{2m-2k}^{(2m)}/\binom{2m-1}{2k} & \text{for } 0\leq k\leq m-1.\end{cases} \tag{17}$$

Inserting this into (6) and (7) yields for even N

$$S_{2m}(N)=\frac{4^{m-1}}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(-1)^{\ell-1}}{\ell}b_{2\ell}^{(2m)}B_{2\ell}N^{2\ell}-\frac{4^{m-1}}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(-1)^{\ell-1}}{\ell}b_{2\ell}^{(2m)}B_{2\ell}-\frac{1}{2} \tag{18}$$

and for odd N

$$S_{2m}(N)=\frac{4^{m-1}}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(-1)^{\ell-1}}{\ell}b_{2\ell}^{(2m)}B_{2\ell}(4^\ell-1)N^{2\ell}-\frac{1}{2} \tag{19}$$

Similarly, we obtain

$$C_{2m}(N)=\frac{4^{m-1}}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(-1)^{\ell-1}}{\ell}b_{2\ell}^{(2m)}B_{2\ell}N^{2\ell}\\ -\frac{4^{m-1}}{(2m-1)!}\sum_{\ell=1}^{m}\frac{(-1)^{\ell-1}}{\ell}b_{2\ell}^{(2m)}B_{2\ell}-\frac{1+(-1)^N}{4}. \tag{20}$$

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