FORMULAS FOR FIBONOMIAL SUMS WITH GENERALIZED FIBONACCI AND LUCAS COEFFICIENTS

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Abstract. We consider certain Fibonomial sums with generalized Fibonacci and Lucas numbers coefficients and compute them explicitly. Some corollaries are also presented. The technique is to rewrite everything in terms of a variable q, and then to use Rothe's identity from classical q-calculus.

1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2,\ V_1 = p.$$

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_U = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$.

Our approach will be as follows. We will use the Binet form

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} \left(1 + xq^k\right),$$

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and *Rothe's* formula [1] is

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1}\left(1-xq^k\right).$$

All the identities we will derive hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q. We will frequently denote $\left\{ {n \atop k} \right\}_U$ by $\left\{ {n \atop k} \right\}$.

We shall consider some Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients, and then we compute these sums by using Rothe's formula after having converted them into forms involving the Gaussian q-binomial coefficients. Some special cases of these sums are also given as corollaries.

Throughout this paper, we will present and prove our main result:

Theorem 1. *If n and m are both nonnegative or are both negative integers, then*

(1)
$$\sum_{k=0}^{2n}\left\{ {2n \atop k} \right\} U_{(2m-1)k} = P_{n,m}\sum_{k=1}^{m}\left\{ {2m-1 \atop 2k-1} \right\} U_{(4k-2)n},$$

(2)
$$\sum_{k=0}^{2n+1}\left\{ {2n+1 \atop k} \right\} U_{2mk} = P_{n,m}\sum_{k=0}^{m}\left\{ {2m \atop 2k} \right\} U_{(2n+1)2k},$$

(3)
$$\sum_{k=0}^{2n}\left\{ {2n \atop k} \right\}(-1)^k U_{(2m-1)k} = P_{n,m}\sum_{k=0}^{m-1}\left\{ {2m-1 \atop 2k} \right\} U_{4kn},$$

(4)
$$\sum_{k=0}^{2n+1}\left\{ {2n+1 \atop k} \right\}(-1)^k U_{2mk} = -P_{n,m}\sum_{k=1}^{m}\left\{ {2m \atop 2k-1} \right\} U_{(2n+1)(2k-1)},$$

where

$$P_{n,m} = \begin{cases} \prod_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m. \end{cases}$$

Proof. First suppose that $n \geq m$. We rewrite $P_{n,m}$ in terms of q-binomial coefficients:

$$\begin{aligned} P_{n,m} &= \prod_{k=0}^{n-m} V_{2k} = \prod_{k=0}^{n-m}\left(\alpha^{2k}+\beta^{2k}\right) \\ &= \alpha^{(n-m)(n-m+1)}\prod_{k=0}^{n-m}\left(1+q^{2k}\right) = 2\alpha^{(n-m)(n-m+1)}(-q^2;q^2)_{n-m} \\ &= 2(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}. \end{aligned}$$

This formula holds for $n < m$ as well, with the usual extension of $(q;q)_n$ to arbitrary n.

Similarly, the first formula takes the following form in terms of q-binomial coefficients:

$$\sum_{k=0}^{2n} \frac{\alpha^{(2m-1)k} - \beta^{(2m-1)k}}{\alpha - \beta} \alpha^{k(2n-k)} \begin{bmatrix} 2n \\ k \end{bmatrix}_q$$
$$= 2\alpha^{(n-m)(n-m+1)}(-q^2;q^2)_{n-m}$$
$$\times \sum_{k=1}^{m} \frac{\alpha^{(4k-2)n} - \beta^{(4k-2)n}}{\alpha - \beta} \alpha^{(2k-1)(2m-1-2k+1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q,$$

which is equivalent to

$$\sum_{k=0}^{2n} [1 - q^{(2m-1)k}] \alpha^{2(m+n)k - 2\binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q$$
$$= 2\alpha^{(n-m)(n-m+1)-2(m+n)}(-q^2;q^2)_{n-m}$$
$$\times \sum_{k=1}^{m} [1 - q^{(4k-2)n}] \alpha^{4k(m+n)-2k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q,$$

and to

$$\sum_{k=0}^{2n} [1 - q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q$$
$$= 2(-q)^{-\binom{n-m+1}{2}+(m+n)}(-q^2;q^2)_{n-m}$$
$$\times \sum_{k=1}^{m} [1 - q^{(4k-2)n}](-1)^k q^{-2k(m+n)+k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q.$$

If we denote the left and right hand sides of this equation by L and R, respectively, then L is the sum of the following two parts:

$$\begin{aligned}
L_1 &= \sum_{k=0}^{2n} (-q)^{-(m+n)k+\binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= \sum_{k=0}^{2n} (-1)^{-(m+n-\frac{1}{2})k+\frac{k^2}{2}} q^{-(m+n-1)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= \sum_{k=0}^{2n} (-1)^{-(m+n-\frac{1}{2})k} q^{-(m+n-1)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \Big[\frac{1+\mathbf{i}}{2} + \frac{1-\mathbf{i}}{2}(-1)^k\Big] \\
&= \frac{1+\mathbf{i}}{2} \sum_{k=0}^{2n} (-1)^{-(m+n-\frac{1}{2})k} q^{-(m+n-1)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&\quad + \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n} (-1)^{-(m+n+\frac{1}{2})k} q^{-(m+n-1)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= \frac{1+\mathbf{i}}{2} (\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} + \frac{1-\mathbf{i}}{2} (-\mathbf{i}(-q)^{-(m+n-1)};q)_{2n}
\end{aligned}$$

and

$$\begin{aligned}
L_2 &= -\sum_{k=0}^{2n} q^{(2m-1)k}(-q)^{-(m+n)k+\binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= -\sum_{k=0}^{2n} (-1)^{(m-n+\frac{1}{2})k+\frac{k^2}{2}} q^{(m-n)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= -\sum_{k=0}^{2n} (-1)^{(m-n+\frac{1}{2})k} q^{(m-n)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \Big[\frac{1+\mathbf{i}}{2} + \frac{1-\mathbf{i}}{2}(-1)^k\Big] \\
&= -\frac{1+\mathbf{i}}{2} \sum_{k=0}^{2n} (-1)^{(m-n+\frac{1}{2})k} q^{(m-n)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&\quad - \frac{1-\mathbf{i}}{2} \sum_{k=0}^{2n} (-1)^{(m-n-\frac{1}{2})k} q^{(m-n)k+\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \\
&= -\frac{1+\mathbf{i}}{2}(-\mathbf{i}(-q)^{m-n}; q)_{2n} - \frac{1-\mathbf{i}}{2}(\mathbf{i}(-q)^{m-n}; q)_{2n}.
\end{aligned}$$

By combining the two parts above we write L as

$$\begin{aligned}
&\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)}; q)_{2n} + \frac{1-\mathbf{i}}{2}(-\mathbf{i}(-q)^{-(m+n-1)}; q)_{2n} \\
&\quad - \frac{1+\mathbf{i}}{2}(-\mathbf{i}(-q)^{m-n}; q)_{2n} - \frac{1-\mathbf{i}}{2}(\mathbf{i}(-q)^{m-n}; q)_{2n} \\
&= L_a + L_b + L_c + L_d.
\end{aligned}$$

Let

$$\begin{aligned}
R_1 &= \sum_{k=1}^{m} [1 - q^{(4k-2)n}](-1)^k q^{-2k(m+n)+k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q \\
&= \mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1} [1 - q^{2kn}]\mathbf{i}^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \frac{1-(-1)^k}{2} \\
&= \frac{1}{2}\mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1} [1 - q^{2kn}]\mathbf{i}^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \\
&\quad - \frac{1}{2}\mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1} [1 - q^{2kn}](-\mathbf{i})^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \\
&= \frac{1}{2}\mathbf{i}q^{-(m+n)}(-\mathbf{i}q^{-m-n+1}; q)_{2m-1} - \frac{1}{2}\mathbf{i}q^{-(m+n)}(-\mathbf{i}q^{-m+n+1}; q)_{2m-1} \\
&\quad - \frac{1}{2}\mathbf{i}q^{-(m+n)}(\mathbf{i}q^{-m-n+1}; q)_{2m-1} + \frac{1}{2}\mathbf{i}q^{-(m+n)}(\mathbf{i}q^{-m+n+1}; q)_{2m-1}.
\end{aligned}$$

In order to form the right hand side R, the last expression must be multiplied by

$$2(-q)^{-\binom{n-m+1}{2}+(m+n)}(-q^2; q^2)_{n-m}.$$

Thus R takes the form:

$$\begin{aligned}
R &= \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1}\\
&\quad - \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m+n+1};q)_{2m-1}\\
&\quad - \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1}\\
&\quad + \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m+n+1};q)_{2m-1}\\
&= R_a + R_b + R_c + R_d.
\end{aligned}$$

We will show that for $m \equiv n \pmod 2$,

$$L_a = R_a, \quad L_b = R_c, \quad L_c = R_b, \quad L_d = R_d,$$

and for $m \not\equiv n \pmod 2$,

$$L_a = R_c, \quad L_b = R_a, \quad L_c = R_d, \quad L_d = R_b.$$

We start with the instance $m \equiv n \pmod 2$, and first show that $L_a = R_a$. If we rearrange both sides of it, the claimed equality becomes

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{-m-n+1};q)_{2n} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{m-n};q)_{2n-2m+1} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}.$$

In order to show the last equality, we consider two cases. For even N, by rearranging both sides of it we get

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{-N};q)_{2N+1} = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2N}(1+\mathbf{i}q^{-N+k}) = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=1}^{N}(1+\mathbf{i}q^{-k})\prod_{k=1}^{N}(1+\mathbf{i}q^{k})(1+\mathbf{i}) = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\prod_{k=1}^{N}(1+\mathbf{i}q^{-k})\prod_{k=1}^{N}(1+\mathbf{i}q^{k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\prod_{k=1}^{N}\mathbf{i}(q^{-k}+q^{k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\mathbf{i}^N q^{-\binom{N+1}{2}}\prod_{k=1}^{N}(1+q^{2k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N$$

or

$$\mathbf{i}^N = (-1)^{N/2} = (-1)^{-(N+1)\frac{N}{2}},$$

as claimed.

Now we prove the second claim $L_b = R_c$. By rearranging both sides of it, we get

$$\frac{1-\mathbf{i}}{2}(-\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1-\mathbf{i}}{2}(\mathbf{i}q^{-m-n+1};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1-\mathbf{i}}{2}(\mathbf{i}q^{m-n};q)_{2n-2m+1} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1-\mathbf{i}}{2}\prod_{k=0}^{2n-2m}(1-\mathbf{i}q^{m-n+k}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1-\mathbf{i}}{2}\prod_{k=1}^{n-m}(1-\mathbf{i}q^{-k})(1-\mathbf{i}q^{k})\cdot(1-\mathbf{i}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$(-\mathbf{i})^{n-m}\prod_{k=1}^{n-m}q^{-k}(1+q^{2k}) = (-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

which becomes

$$\mathbf{i}^{n-m} = (-1)^{-\binom{n-m+1}{2}},$$

as claimed.

We note that the other cases (for $m \equiv n \pmod 2$) can be done similarly.

Now we consider the case $L_a = R_c$ if $m \not\equiv n \pmod 2$. By simplifying both sides of the claimed equality step by step, we get

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1}$$

or

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}q^{-m-n+1};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1}$$

or

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}q^{m-n};q)_{2n-2m+1} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2n-2m}(1-\mathbf{i}q^{m-n+k}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=1}^{n-m}(1-\mathbf{i}q^{-k})(1-\mathbf{i}q^{k})\cdot(1-\mathbf{i}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}$$

or

$$\mathbf{i}^{n-m}\prod_{k=1}^{n-m}q^{-k}(1+q^{2k}) = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}$$

or

$$\mathbf{i}^{n-m} = \mathbf{i}(-1)^{-\binom{n-m+1}{2}},$$

which is true as claimed.

The other cases (for $m \not\equiv n \pmod 2$) can be done similarly.

The arguments hold for $n < m$ as well.

The rest of claimed identities can be proved in the same style, with only minor variations. ∎

Theorem 2. *If n and m are both nonnegative or are both negative integers, then*

(1)
$$\sum_{k=0}^{2n}\left\{ 2n \atop k \right\}V_{(2m-1)k}=P_{n,m}\sum_{k=1}^{m}\left\{ 2m-1 \atop 2k-1 \right\}V_{(4k-2)n},$$

(2)
$$\sum_{k=0}^{2n+1}\left\{ 2n+1 \atop k \right\}V_{2mk}=P_{n,m}\sum_{k=0}^{m}\left\{ 2m \atop 2k \right\}V_{(2n+1)2k},$$

(3)
$$\sum_{k=0}^{2n}\left\{ 2n \atop k \right\}(-1)^k V_{(2m-1)k}=P_{n,m}\sum_{k=0}^{m-1}\left\{ 2m-1 \atop 2k \right\}V_{4kn},$$

(4)
$$\sum_{k=0}^{2n+1}\left\{ 2n+1 \atop k \right\}(-1)^k V_{2mk}=-P_{n,m}\sum_{k=1}^{m}\left\{ 2m \atop 2k-1 \right\}V_{(2n+1)(2k-1)},$$

where $P_{n,m}$ is defined as before.

Proof. The proofs of the claimed identities can be done similarly to the proof of Theorem 1. ∎

For example, when $m = n$ in Theorem 1, we have the following identities:

(1)
$$\sum_{k=0}^{2n}\left\{ 2n \atop k \right\}U_{(2n-1)k}=2\sum_{k=1}^{n}\left\{ 2n-1 \atop 2k-1 \right\}U_{(4k-2)n},$$

(2)
$$\sum_{k=0}^{2n+1}\left\{ 2n+1 \atop k \right\}U_{2nk}=2\sum_{k=0}^{n}\left\{ 2n \atop 2k \right\}U_{(2n+1)2k},$$

(3)
$$\sum_{k=0}^{2n}\left\{ 2n \atop k \right\}(-1)^k U_{(2n-1)k}=2\sum_{k=0}^{n-1}\left\{ 2n-1 \atop 2k \right\}U_{4kn},$$

(4)
$$\sum_{k=0}^{2n+1}\left\{ 2n+1 \atop k \right\}(-1)^k U_{2nk}=-2\sum_{k=1}^{n}\left\{ 2n \atop 2k-1 \right\}U_{(2n+1)(2k-1)}.$$

For the reader's convenience, here is the complete list of q-binomial versions of the identities given in Theorem 1 and Theorem 2: Let n and m be both nonnegative or both negative integers,

$$\sum_{k=0}^{2n}[1-q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q = P_{n,m}\sum_{k=1}^{m}[1-q^{(4k-2)n}](-q)^{-(2k-1)(m+n-k)}\begin{bmatrix}2m-1\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}[1-q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q = P_{n,m}\sum_{k=0}^{m}[1-q^{2k(2n+1)}](-q)^{k(2k-2m-2n-1)}\begin{bmatrix}2m\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}(-1)^k[1-q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q = P_{n,m}\sum_{k=0}^{m-1}[1-q^{4kn}](-q)^{k(2k-2m-2n+1)}\begin{bmatrix}2m-1\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}(-1)^k[1-q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q = -P_{n,m}\sum_{k=1}^{m}[1-q^{(2k-1)(2n+1)}](-q)^{(2k-1)(k-m-n-1)}\begin{bmatrix}2m\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}[1+q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q = P_{n,m}\sum_{k=1}^{m}[1+q^{(4k-2)n}](-q)^{-(2k-1)(m+n-k)}\begin{bmatrix}2m-1\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}[1+q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q = P_{n,m}\sum_{k=0}^{m}[1+q^{2k(2n+1)}](-q)^{k(2k-2m-2n-1)}\begin{bmatrix}2m\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}(-1)^k[1+q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q = P_{n,m}\sum_{k=0}^{m-1}[1+q^{4kn}](-q)^{k(2k-2m-2n+1)}\begin{bmatrix}2m-1\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}(-1)^k[1+q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$= -P_{n,m}\sum_{k=1}^{m}[1+q^{(2k-1)(2n+1)}](-q)^{(2k-1)(k-m-n-1)}\begin{bmatrix}2m\\2k-1\end{bmatrix}_q,$$

where

$$P_{n,m} = \begin{cases} 2(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m} & \text{if } n \geq m, \\ (-q)^{\binom{m-n}{2}}(-q^2;q^2)^{-1}_{m-n-1} & \text{if } n < m. \end{cases}$$

Remark. It is not necessary to split the definition of $P_{n,m}$, as the first alternative would work in both cases, but it is more convenient as given.

References

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