THE INTRINSIC PERIODIC BEHAVIOUR OF SEQUENCES RELATED TO A RATIONAL INTEGRAL

With corrections suggested by Jean-Paul Allouche

Helmut Prodinger[1]
Mathematics Department, Stellenbosch University, 7602 Stellenbosch, South Africa
hproding@sun.ac.za

Abstract
For sequences defined in terms of 2-adic valuations, we exploit the intrinsic periodic behaviour obtained by a double summation. The tool is the Mellin-Perron formula.

1. Introduction

In [2], Sun and Moll studied various 2-adic valuations related to a certain integral. The purpose of this note is to illustrate this somewhat, by relating them to periodic oscillations that are common when studying the sum of digits function.

The 2-adic valuation $\nu_2(n)$ is the largest power of 2 that divides n, and satisfies the recursion formula $\nu_2(2n+1) = 0$, $\nu_2(2n) = 1 + \nu_2(n)$.

It is known (see, e. g., [1]), that $S_2(n) = n - \sum_{0 \le k \le n} \nu_2(k)$, where $S_2(n)$ is the sum of digits (=ones) in the binary representation of n. This function itself is still quite erratic, but the summatory function $\sum_{k<n} S_2(k)$ is nice and smooth, and possesses a representation as $\frac{1}{2} n \log_2 n + n\delta(\log_2 n)$, with a periodic function $\delta(x)$. We refer for all this to [1] and the references provided therein.

Let us start with the simplest example from [2]:

$$f_3(m) = \begin{cases} 7 + \nu_2(\frac{m+1}{2}) & \text{if } m \equiv 1 \bmod 2, \\ 9 + \nu_2(\frac{m}{4}) & \text{if } m \equiv 0 \bmod 4, \\ 9 + \nu_2(\frac{m+2}{4}) & \text{if } m \equiv 2 \bmod 4. \end{cases}$$

So we expect that a double summation over $f_3(m)$ will result in attractive results. This and another example from [2] will be considered in the sequel.

[1]The author is supported by an incentive grant from the NRF (South Africa).

2. Mellin-Perron approach

A double summation can be expressed as a single sum via

$$\sum_{1\le k<n} f_3(k)(n-k) = n\sum_{1\le k<n} f_3(k)\Big(1-\frac{k}{n}\Big).$$

The Mellin-Perron approach, as developed in [1], provides an integral representation for it:

$$\sum_{1\le k<n} f_3(k)\Big(1-\frac{k}{n}\Big) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_3(s)n^s\frac{ds}{s(s+1)},$$

where

$$F_3(s) = \sum_{n\ge 1}\frac{f_3(n)}{n^s}$$

is the generating Dirichlet series, and c is any real number large enough, so that the series converges.

$$\begin{aligned}
F_3(s) &= \sum_{n\ge 1}\frac{f_3(n)}{n^s}\\
&= \sum_{n\ge 0}\frac{7+\nu_2(n+1)}{(2n+1)^s} + \sum_{n\ge 1}\frac{9+\nu_2(n)}{(4n)^s} + \sum_{n\ge 0}\frac{9+\nu_2(n+1)}{(4n+2)^s}\\
&= 9\zeta(s) - 2\sum_{n\ge 0}\frac{1}{(2n+1)^s} + \sum_{n\ge 0}\frac{\nu_2(n+1)}{(2n+1)^s} + \sum_{n\ge 1}\frac{\nu_2(n)}{(4n)^s} + \sum_{n\ge 0}\frac{\nu_2(n+1)}{(4n+2)^s}\\
&= 9\zeta(s) - 2\zeta(s) + 2^{1-s}\zeta(s) + (1+2^{-s})\sum_{n\ge 0}\frac{\nu_2(n+1)}{(2n+1)^s} + 4^{-s}\sum_{n\ge 1}\frac{\nu_2(n)}{n^s}.
\end{aligned}$$

Now we compute (as many people did before)

$$\sum_{n\ge 1}\frac{\nu_2(n)}{n^s} = \sum_{n\ge 1}\frac{1+\nu_2(n)}{(2n)^s}.$$

So

$$\sum_{n\ge 1}\frac{\nu_2(n)}{n^s} = \frac{2^{-s}\zeta(s)}{1-2^{-s}}.$$

The remaining sum we treat like this:

$$\begin{aligned}\sum_{n\ge1}\frac{\nu_2(n)}{(2n-1)^s}&=\sum_{n\ge1}\frac{\nu_2(n)}{(2n)^s}\Big(1-\frac{1}{2n}\Big)^{-s}\\&=\sum_{k\ge0}\binom{s+k-1}{k}\sum_{n\ge1}\frac{\nu_2(n)}{(2n)^{s+k}}\\&=\sum_{k\ge0}\binom{s+k-1}{k}\frac{1}{2^{s+k}(2^{s+k}-1)}\zeta(s+k).\end{aligned}$$

Summarizing:

$$\begin{aligned}F_3(s)&=7\zeta(s)+2^{1-s}\zeta(s)+\frac{8^{-s}\zeta(s)}{1-2^{-s}}\\&\quad+(1+2^{-s})\sum_{k\ge0}\binom{s+k-1}{k}\frac{1}{2^{s+k}(2^{s+k}-1)}\zeta(s+k).\end{aligned}$$

The classic procedure is now to shift the line of integration to the left and collect residues. Because of the presence of the zeta function, we cannot shift the line too far; and because of the functions $\zeta(s+k)$, we cannot hope for an *exact* formula, but only for an asymptotic one. More details about this procedure are to be found in the article [1].

We must collect the residues of

$$F_3(s)\frac{n^s}{s(s+1)}$$

at $s=1$ and at $s=0$. Fortunately, this can be done by a computer, and we have collected so far:

$$\frac{9n}{2}-\frac{3}{2}\log_2 n-\frac{7}{4}-\frac{3}{2}\log_2\pi+\frac{3}{2\log 2}.$$

But there are also poles at $s=\chi_k=\frac{2\pi ik}{\log 2}$, with residues

$$\frac{3\zeta(\chi_k)}{\log 2\cdot\chi_k(\chi_k+1)}e^{2\pi ik\cdot\log_2 n}.$$

Traditionally, one collects them into a *periodic function*:

$$\phi(x)=\frac{3}{\log 2}\sum_{k\ne0}\frac{\zeta(\chi_k)}{\chi_k(\chi_k+1)}e^{2\pi ikx}.$$

Multiplying all the contributions by n, we found an asymptotic expansion for the doubly iterated sum. The remainder term stems from the fact that we shift the line of integration to $\Re s=-\frac{1}{4}$, as in [1].

Theorem 1

$$\sum_{1\le k<n} f_3(k)(n-k) = \frac{9n^2}{2} - \frac{3n}{2}\log_2 n - \frac{3n}{2}\log_2 \pi - \frac{7n}{4} + \frac{3n}{2\log 2} + n\phi(\log_2 n) + O(n^{3/4}).$$

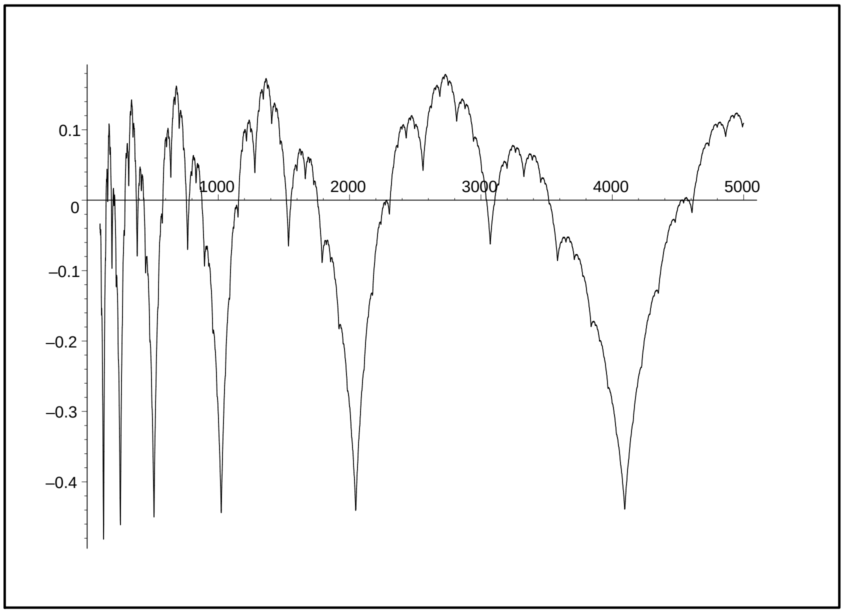

Figure 1: $\frac{1}{n}\sum_{1\le k<n} f_3(k)(n-k) - \frac{9n}{2} + \frac{3}{2}\log_2 n + \frac{7}{4} + \frac{3}{2}\log_2 \pi - \frac{3}{2\log 2}$

Now let us move to the next example from [2]:

$$f_5(m) = \begin{cases} 14 + \nu_2(\frac{m+2}{4}) & \text{if } m \equiv 2 \bmod 4, \\ 13 + \nu_2(\frac{m+1}{4}) & \text{if } m \equiv 3 \bmod 4, \\ 13 + \nu_2(\frac{m+3}{4}) & \text{if } m \equiv 1 \bmod 4, \\ 16 + \nu_2(\frac{m}{8}) & \text{if } m \equiv 0 \bmod 8, \\ 16 + \nu_2(\frac{m+4}{8}) & \text{if } m \equiv 4 \bmod 8. \end{cases}$$

As before, we compute the generating Dirichlet series:

$$\sum_{n\ge1} \frac{f_5(n)}{n^s} = 16\zeta(s) + \sum_{n\ge0} \frac{-2+\nu_2(n+1)}{(4n+2)^s} + \sum_{n\ge0} \frac{-3+\nu_2(n+1)}{(4n+3)^s} + \sum_{n\ge0} \frac{-3+\nu_2(n+1)}{(4n+1)^s} + \sum_{n\ge1} \frac{\nu_2(n)}{(8n)^s} + \sum_{n\ge0} \frac{\nu_2(n+1)}{(8n+4)^s}$$

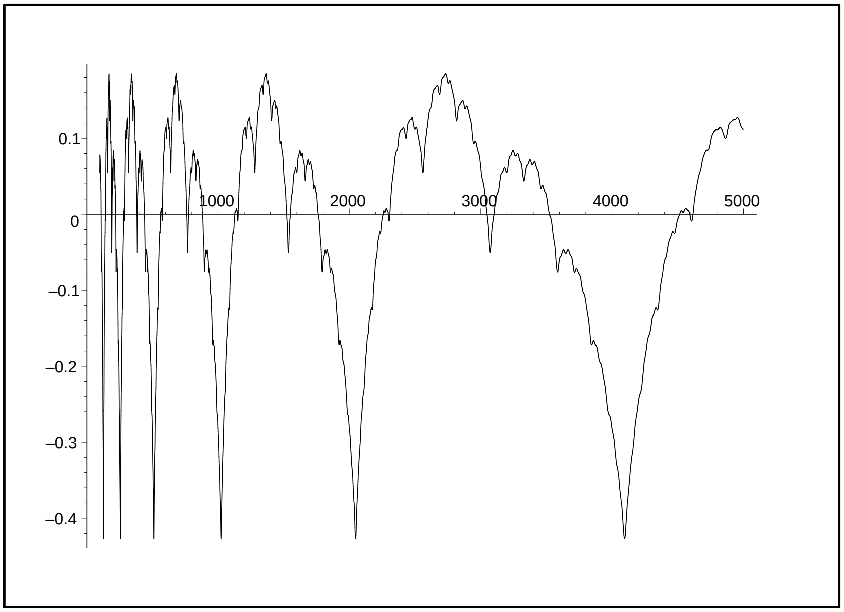

Figure 2: The periodic function $\phi(\log_2 n)$, drawn from the first 200 Fourier coefficients.

$$
\begin{aligned}
&= 16\zeta(s) - 3\zeta(s) + 2\,4^{-s}\zeta(s) + 2^{-s}\zeta(s) + \sum_{n\ge1}\frac{\nu_2(n)}{(4n-2)^s} + \sum_{n\ge1}\frac{\nu_2(n)}{(4n-1)^s}\\
&\quad + \sum_{n\ge1}\frac{\nu_2(n)}{(4n-3)^s} + 8^{-s}\sum_{n\ge1}\frac{\nu_2(n)}{n^s} + \sum_{n\ge1}\frac{\nu_2(n)}{(8n-4)^s}\\
&= 13\zeta(s) + 2\,4^{-s}\zeta(s) + 2^{-s}\zeta(s) + (2^{-s}+4^{-s})\sum_{n\ge1}\frac{\nu_2(n)}{(2n-1)^s}\\
&\quad + \sum_{n\ge1}\frac{\nu_2(n)}{(4n-1)^s} + \sum_{n\ge1}\frac{\nu_2(n)}{(4n-3)^s} + \frac{16^{-s}}{1-2^{-s}}\zeta(s)\\
&= 13\zeta(s) + 2\,4^{-s}\zeta(s) + 2^{-s}\zeta(s) + \frac{16^{-s}}{1-2^{-s}}\zeta(s)\\
&\quad + (2^{-s}+4^{-s})\sum_{k\ge0}\binom{s+k-1}{k}\frac{1}{2^{s+k}(2^{s+k}-1)}\zeta(s+k)\\
&\quad + \sum_{k\ge0}\binom{s+k-1}{k}\frac{1+3^k}{4^{s+k}(2^{s+k}-1)}\zeta(s+k).
\end{aligned}
$$

The collected residues of $\dfrac{F_5(s)n^s}{s(s+1)}$ at $s=1$ and $s=0$ are:

$$\frac{15n}{2} - \frac{5}{2}\log_2 n - \frac{5}{2}\log_2 \pi - \frac{5}{4} + \frac{5}{2\log 2}.$$

Theorem 2

$$\sum_{1\le k<n} f_5(k)(n-k) = \frac{15n^2}{2} - \frac{5n}{2}\log_2 n - \frac{5n}{2}\log_2 \pi - \frac{5n}{4} + \frac{5n}{2\log 2} + n\psi(\log_2 n) + O(n^{3/4}).$$

Other examples, related to $f_7(m)$, $f_9(m)$, $\ldots$, can be treated in the same style, but the generating Dirichlet series become more cumbersome.

References

[1] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy. Mellin transforms and asymptotics: Digital sums. *Theoretical Computer Science*, 123:291–314, 1994.

[2] X. Sun and V. H. Moll. A binary tree representation for the 2-adic valuation of a sequence arising from a rational integral. *Integers*, 10:211–222, 2010.