THE INVERSE FILBERT MATRIX

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Abstract. The Filbert matrix $\mathcal{F}_N$ has entries $\frac{1}{F_{i+j-1}}$ and is an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. Various extensions are nowadays known with beautiful explicit results (LU-decomposition, Cholesky decompositions, etc.). The *inverse* Filbert matrix does, in these general instances, *not* have closed form entries. Nevertheless, it can be decomposed *itself* in very much the same way as the matrix. These results are *not* corollaries of the decomposition of the Filbert matrix itself.

1. Introduction

The Filbert matrix $\mathcal{F}_N = (\check{h}_{ij})_{i,j=1}^N$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where F_n is the nth Fibonacci number. It has been defined and studied by Richardson [9] and extended by Berg [1] and Ismail [2].

Kilic and Prodinger, in a series of papers [3, 7, 4, 5, 6], extended the concept to matrices with entries

$$\frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}} \quad \text{and} \quad \frac{1}{L_{\lambda(i+j)+r}L_{\lambda(i+j+1)+r}\dots L_{\lambda(i+j+k-1)+r}}.$$

Here, $\lambda, k \ge 1$ and $r \ge -1$ are integer parameters.

These generalizations were driven by the search for "nice" explicit formulæ for

- LU-decomposition of the matrix M as $M = LU$,
- explicit description of L^{-1}, U^{-1},
- the Cholesky decomposition $M = C \cdot C^T$,
- the inverse matrix M^{-1}.

Now, for $k \ge 2$, the inverse matrix (both, in the Fibonacci and Lucas instances $\mathcal{F}_N$ and $\mathcal{L}_N$) are no longer nice; the entries can only be given as a (single) sum which cannot be simplified.

Therefore, in this paper, we took the inverse matrices $\mathcal{F}_N^{-1}$ and $\mathcal{L}_N^{-1}$ as the focus of our attention. It came somewhat as a surprise that LU-decompositions AB of these matrices led to nice (=closed form) results, with A^{-1}, B^{-1}, as well as the Cholesky decomposition $D \cdot D^T$ and its inverse D^{-1} also being nice! Note carefully that from two LU-decompositions $M = LU$ and $M^{-1} = AB$, there is no obvious way to link the matrices A and B to L and U. Furthermore, all these matrices appearing in our new decompositions *depend* on the dimension N. This is in sharp contrast to the "old" cases where one could always think about *one* infinite matrix, and restricts oneself to the first N rows and columns.

The second author was supported by an incentive grant of the NRF of South Africa.

In another direction [5], the matrices $\mathcal{G}_N$ and $\mathcal{V}_N$ with entries

$$g_{ij} = \frac{F_{\lambda(i+j)+r}}{F_{\lambda(i+j)+s}} \quad \text{and} \quad v_{ij} = \frac{L_{\lambda(i+j)+r}}{L_{\lambda(i+j)+s}}$$

were introduced; here s, r and λ are integer parameters such that $s \neq r$, and $r, s \geq -1$ and $\lambda \geq 1$. For this situation, it seems to be impossible to introduce an extra parameter k as above, in order to get reasonable results.

We managed the decompose the inverse matrices $\mathcal{G}_N^{-1}$ and $\mathcal{V}_N^{-1}$ as well. These inverse matrices are of closed form entries this time, but it is interesting anyway to study their decompositions.

Now we discuss our settings. Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas sequences, respectively, whose the Binet forms are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad L_n = \alpha^n + \beta^n = \alpha^n(1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

We will exclusively deal with the q-forms; translating the results back to the Fibonacci and Lucas world is easy: We only have to systematically replace $1-q^n$ by $\frac{1-q}{\alpha^{n-1}}F_n$ and $1+q^n$ by $\alpha^n L_n$ and replace what is eventually left by its numerical values. One might also think about *one parameter extensions* of Fibonacci resp. Lucas numbers.

Throughout this paper we will use the notation of the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$.

The important contribution of this paper is to *find* the explicit forms of the various entries. This was done by experiments with a computer algebra system and spotting patterns. This becomes increasingly complicated when more and more new parameters are introduced, as the guessing only works for fixed choices of the parameters, and one needs to vary them as well.

Once one *knows* how the entries look like, proofs are by reducing sums to single terms. For this, the q-Zeilberger algorithm is a handy tool. In some instances, this does not work, and we have to *simulate* the q-Zeilberger algorithm *manually* by doing more guessing (with an additional parameter).

These proofs are routine and somewhat tedious; we will thus only present a few of them and leave the others to the imagination of the reader.

Instead of proving that $AB = M^{-1}$ where M^{-1} has ugly coefficients, we prove the equivalent statement $B^{-1}A^{-1} = M$. Of course, that requires first to prove that the forms given for A^{-1} and B^{-1} are indeed correct.

2. The matrix $\mathcal{F}_N^{-1}$

We consider the matrix $\mathcal{F}_N$ with entries

$$\frac{1}{F_{\lambda(n+d)+r}F_{\lambda(n+d+1)+r}\cdots F_{\lambda(n+d+k-1)+r}}$$

for $1 \le n, d \le N$. First, it is easy to check that the entries in rewritten form are

$$q^{\lambda\frac{kn}{2}+\lambda\frac{kd}{2}+\lambda\frac{k(k-1)}{4}-\frac{k}{2}+\frac{rk}{2}}\mathbf{i}^{-\lambda kn-\lambda kd-\lambda\frac{k(k-1)}{2}-rk+k}\frac{(q^{\lambda+r};q^{\lambda})_{n+d-1}(1-q)^k}{(q^{\lambda+r};q^{\lambda})_{n+d+k-1}}.$$

We denote the LU-decomposition and Cholesky-decomposition by

$$\mathcal{F}_N^{-1} = A \cdot B = D \cdot D^T.$$

Then we get the following results:

Theorem 1. *For $1 \le d \le n \le N$:*

- $$A_{n,d} = q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d+1)}{2}+\lambda N(d-n)+\lambda\frac{k(d-n)}{2}}\mathbf{i}^{\lambda(n-d)k+2(n-d)} \times \frac{(q^{\lambda+r};q^{\lambda})_{N+n+k-1}(q^{\lambda};q^{\lambda})_{N-d}(q^{\lambda+r};q^{\lambda})_{2d}}{(q^{\lambda+r};q^{\lambda})_{N+d+k-1}(q^{\lambda+r};q)_{n+d}(q^{\lambda};q^{\lambda})_{n-d}(q^{\lambda};q^{\lambda})_{N-n}}$$
- $$A^{-1}_{n,d} = q^{-\lambda Nn+\lambda n^2+\lambda Nd-\lambda nd-\lambda\frac{kn}{2}+\lambda\frac{kd}{2}}\mathbf{i}^{\lambda k(n-d)} \times \frac{(q^{\lambda+r};q^{\lambda})_{N+n+k-1}(q^{\lambda};q^{\lambda})_{N-d}(q^{\lambda+r};q^{\lambda})_{n-1+d}}{(q^{\lambda+r};q^{\lambda})_{N+d+k-1}(q^{\lambda};q^{\lambda})_{N-n}(q^{\lambda+r};q^{\lambda})_{2n-1}(q^{\lambda};q^{\lambda})_{n-d}}$$
- $$B_{d,n} = q^{\lambda\frac{n(n+1)}{2}-\lambda Nn+\lambda\frac{3d^2}{2}-\lambda Nd-\lambda\frac{d}{2}-\lambda\frac{kn}{2}-\lambda\frac{kd}{2}-\lambda\frac{k(k-1)}{4}-\frac{kr}{2}-rN+\frac{k}{2}+rd} \times \mathbf{i}^{\lambda nk+\lambda dk-2n-2d+\lambda\frac{k(k-1)}{2}+kr-k} \times \frac{(q^{\lambda+r};q^{\lambda})_{N+n+k-1}(q^{\lambda+r};q^{\lambda})_{N+d}(q^{\lambda};q^{\lambda})_{k-1}}{(q^{\lambda};q^{\lambda})_{N-d+k-1}(q^{\lambda};q^{\lambda})_{N-n}(q^{\lambda};q^{\lambda})_{n-d}(q^{\lambda+r};q^{\lambda})_{n+d}(q^{\lambda+r};q^{\lambda})_{2d-1}(1-q)^k}$$
- $$B^{-1}_{d,n} = \mathbf{i}^{-\lambda kn-\lambda kd+\lambda\frac{3k(k-1)}{2}-kr+k} \times q^{\lambda Nn-\lambda nd-\lambda n^2+\lambda Nd+\lambda\frac{kn}{2}+\lambda\frac{kd}{2}+\lambda\frac{k(k-1)}{4}-rn+Nr+\frac{kr}{2}-\frac{k}{2}} \times \frac{(q^{\lambda};q^{\lambda})_{N-n+k-1}(q^{\lambda};q^{\lambda})_{N-d}(q^{\lambda+r};q^{\lambda})_{2n}(q^{\lambda+r};q^{\lambda})_{n+d-1}(1-q)^k}{(q^{\lambda+r};q^{\lambda})_{N+n}(q^{\lambda+r};q^{\lambda})_{N+d+k-1}(q^{\lambda};q^{\lambda})_{n-d}(q^{\lambda};q^{\lambda})_{k-1}}$$
- $$D_{n,d} = q^{\lambda\frac{d^2}{2}-\lambda\frac{d}{2}+\lambda\frac{nk}{2}+\lambda\frac{k(k-1)}{8}+\frac{rd}{2}-\frac{r}{2}+\frac{rk}{4}-\frac{k}{4}}\mathbf{i}^{\lambda kn+k^2-\frac{k}{2}-\frac{rk}{2}-\lambda\frac{k(k-1)}{4}} \times \frac{(q^{\lambda};q^{\lambda})_{n-1}(q^{\lambda+r};q^{\lambda})_n(1-q)^{k/2}}{(q^{\lambda+r};q^{\lambda})_{n+d+k-1}(q^{\lambda};q^{\lambda})_{n-d}} \times \sqrt{\frac{(q^{\lambda};q^{\lambda})_{d+k-2}(q^{\lambda+r};q^{\lambda})_{d+k-1}}{(q^{\lambda};q^{\lambda})_{k-1}(q^{\lambda+r};q^{\lambda})_d(q^{\lambda};q^{\lambda})_{d-1}}(1-q^{\lambda(2d+k-1)+r})}$$

- $$D^{-1}_{n,d} = q^{-\lambda dn+\lambda\frac{d^2}{2}+\lambda\frac{d}{2}-\lambda\frac{dk}{2}-\lambda\frac{k(k-1)}{8}-\frac{rn}{2}+\frac{r}{2}-\frac{rk}{4}+\frac{k}{4}}\mathbf{i}^{-\lambda kd+\lambda\frac{k(k-1)}{4}+\frac{rk}{2}-\frac{k}{2}}$$
$$\times\frac{(q^{\lambda+r};q^\lambda)_{n+d+k-2}}{(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{d-1}(q^{\lambda+r};q^\lambda)_d(1-q)^{k/2}}$$
$$\times\sqrt{\frac{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda+r};q^\lambda)_n(q^\lambda;q^\lambda)_{k-1}(1-q^{\lambda(2n+k-1)+r})}{(q^\lambda;q^\lambda)_{n+k-2}(q^{\lambda+r};q^\lambda)_{n+k-1}}}$$

3. The matrix $\mathcal{L}_N^{-1}$

We consider the matrix $\mathcal{L}_N$ with entries

$$\frac{1}{L_{\lambda(n+d)+r}L_{\lambda(n+d+1)+r}\dots L_{\lambda(n+d+k-1)+r}}$$

for $1\le n,d\le N$.

We denote again the LU-decomposition and Cholesky-decomposition by

$$\mathcal{L}_N^{-1}=A\cdot B=D\cdot D^T;$$

there is no danger of confusion to use the same letters again.

Then we get the following results:

Theorem 2. *For $1\le d\le n\le N$:*

- $$A_{n,d}=q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d+1)}{2}-\lambda N(n-d)-\lambda\frac{k(n-d)}{2}}\mathbf{i}^{\lambda k(n-d)-2(n-d)}$$
$$\times\frac{(-q^{\lambda+r};q^\lambda)_{N+n+k-1}(-q^{\lambda+r};q^\lambda)_{2d}(q^\lambda;q^\lambda)_{N-d}}{(-q^{\lambda+r};q^\lambda)_{N+d+k-1}(-q^{\lambda+r};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{n-d}}$$
- $$A^{-1}_{n,d}=q^{\lambda n^2-\lambda Nn+\lambda Nd-\lambda nd-\lambda\frac{kn}{2}+\lambda\frac{kd}{2}}\mathbf{i}^{\lambda k(n-d)}$$
$$\times\frac{(-q^{\lambda+r};q^\lambda)_{N+n+k-1}(-q^{\lambda+r};q^\lambda)_{n+d-1}(q^\lambda;q^\lambda)_{N-d}}{(-q^{\lambda+r};q^\lambda)_{N+d+k-1}(-q^{\lambda+r};q^\lambda)_{2n-1}(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{n-d}}$$
- $$B_{d,n}=q^{\lambda\frac{n(n+1)}{2}-\lambda Nn+\lambda\frac{d(3d-1)}{2}-\lambda\frac{kn}{2}+\lambda\frac{kd}{2}-\lambda Nd-\lambda kd-\lambda\frac{k(k-1)}{4}+rd-Nr-\frac{kr}{2}}$$
$$\times\mathbf{i}^{\lambda kn-2n+\lambda kd+\lambda\frac{k(k-1)}{2}+2N+rk}$$
$$\times\frac{(-q^{\lambda+r};q^\lambda)_{N+n+k-1}(-q^{\lambda+r};q^\lambda)_{N+d+k-1}(-q^{\lambda+r};q^\lambda)_{N+d}}{(-q^{\lambda+r};q^\lambda)_{N+d+k-1}(-q^{\lambda+r};q^\lambda)_{n+d}(-q^{\lambda+r};q^\lambda)_{2d-1}}$$
$$\times\frac{(q^\lambda;q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{N-d+k-1}}$$
- $$B^{-1}_{d,n}=q^{-\lambda n^2+\lambda Nn-\lambda dn+\lambda Nd+\lambda\frac{kn}{2}+\lambda\frac{dk}{2}+\lambda\frac{k(k-1)}{4}-rn+Nr+\frac{kr}{2}}$$
$$\times\mathbf{i}^{-\lambda k(n+d)-\lambda\frac{k(k-1)}{2}+2N-2n-rk}$$

$$\times \frac{(-q^{\lambda+r};q^\lambda)_{n+d+1}(-q^{\lambda+r};q^\lambda)_{2n}}{(-q^{\lambda+r};q^\lambda)_{N+n}(-q^{\lambda+r};q^\lambda)_{N+d+k-1}}\frac{(q^\lambda;q^\lambda)_{N-n+k-1}(q^\lambda;q^\lambda)_{N-d}}{(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{k-1}}$$

- $$D_{n,d}=q^{\lambda\frac{n(n+1)}{2}-\lambda Nn+\lambda\frac{d(d-1)}{2}-\lambda\frac{kn}{2}-\lambda\frac{k(k-1)}{8}+\frac{rd}{2}-\frac{rk}{4}-\frac{Nr}{2}}$$
$$\times\,\mathbf{i}^{d+N+\lambda\frac{k(k-1)}{4}+\lambda kn+\frac{kr}{2}}\frac{(-q^{\lambda+r};q^\lambda)_{N+n+k-1}}{(-q^{\lambda+r};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{n-d}}$$
$$\times\sqrt{\frac{(q^\lambda;q^\lambda)_{N-d}(-q^{\lambda+r};q^\lambda)_{N+d}(q^\lambda;q^\lambda)_{k-1}(1+q^{2\lambda d+r})}{(q^\lambda;q^\lambda)_{N-d+k-1}(-q^{\lambda+r};q^\lambda)_{N+d+k-1}}}$$

- $$D^{-1}_{n,d}=q^{-\lambda dn+\lambda dN+\lambda\frac{dk}{2}+\lambda\frac{k(k-1)}{8}-\frac{rn}{2}+\frac{rk}{4}+\frac{Nr}{2}}\,\mathbf{i}^{n+\lambda kd+N-\lambda\frac{k(k-1)}{4}-\frac{rk}{2}}$$
$$\times\frac{(-q^{\lambda+r};q^\lambda)_{n+d-1}(q^\lambda;q^\lambda)_{N-d}}{(-q^{\lambda+r};q^\lambda)_{N+d+k-1}(q^\lambda;q^\lambda)_{n-d}}$$
$$\times\sqrt{\frac{(q^\lambda;q^\lambda)_{N-n+k-1}(-q^{\lambda+r};q^\lambda)_{N+n+k-1}(1+q^{2\lambda n+r})}{(q^\lambda;q^\lambda)_{N-n}(-q^{\lambda+r};q^\lambda)_{N+n}(q^\lambda;q^\lambda)_{k-1}}}$$

4. The matrix $\mathcal{G}_N^{-1}$

We consider $\mathcal{G}_N$ with entries

$$\frac{F_{\lambda(n+d)+r}}{F_{\lambda(n+d)+s}}$$

and again

$$\mathcal{G}_N^{-1}=A\cdot B=D\cdot D^T.$$

Theorem 3. *For $1\le d\le n\le N$:*

- $$A_{n,d}=(-1)^{n-d}q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d+1)}{2}+\lambda Nd-\lambda Nn}\frac{1-q^{\lambda N(N+1)+sN+r-(d+1)s-\lambda d^2-\lambda n}}{1-q^{\lambda N(N+1)+sN+r-(d+1)s-\lambda d^2-\lambda d}}$$
$$\times\frac{(q^{s+\lambda};q^\lambda)_{N+n}(q^\lambda;q^\lambda)_{N-d}(q^{s+\lambda};q^\lambda)_{2d}}{(q^\lambda;q^\lambda)_{N-n}(q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}(q^{s+\lambda};q^\lambda)_{N+d}}$$

- $$A^{-1}_{n,d}=q^{\lambda n^2-\lambda Nn+\lambda Nd-\lambda nd}\frac{1-q^{\lambda d+\lambda N(N+1)-\lambda n^2+r+s(N-n)}}{1-q^{\lambda n+\lambda N(N+1)-\lambda n^2+r+s(N-n)}}$$
$$\times\frac{(q^{s+\lambda};q^\lambda)_{n+d-1}(q^{s+\lambda};q^\lambda)_{N+n}(q^\lambda;q^\lambda)_{N-d}}{(q^{s+\lambda};q^\lambda)_{2n-1}(q;q)_{n-d}(q^\lambda;q^\lambda)_{N-n}(q^{s+\lambda};q^\lambda)_{N+d}}$$

- $$B_{d,n}=(-1)^{n-d}\mathbf{i}^{s-r}q^{\lambda\frac{n^2}{2}+\lambda\frac{3d^2}{2}-\lambda Nn-\lambda Nd+\lambda\frac{n-d}{2}+sd-Ns-\frac{s-r}{2}}\frac{1-q^{\lambda N(N+1)-\lambda d^2-\lambda n+r+s(N-d-1)}}{1-q^{\lambda N(N+1)-\lambda d^2+\lambda d+r+s(N-d)}}$$
$$\times\frac{(q^{s+\lambda};q^\lambda)_{N+n}(q^{s+\lambda};q^\lambda)_{N+d}}{(q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{N-n}(q^{s+\lambda};q^\lambda)_{2d-1}(q^\lambda;q^\lambda)_{N-d}}\frac{1}{1-q^{r-s}}$$

- $$B^{-1}_{d,n} = \mathbf{i}^{r-s} q^{-\lambda n^2+\lambda Nn+\lambda Nd-\lambda nd-\frac{r-s}{2}+s(N-n)}(1-q^{r-s}) \times \frac{1-q^{\lambda N(N+1)-\lambda n^2+\lambda d+r+s(N-n)}}{1-q^{\lambda N(N+1)-\lambda n^2-\lambda n+r+s(N-n-1)}} \frac{(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{N-d}(q^{s+\lambda};q^\lambda)_{n+d-1}(q^{s+\lambda};q)_{2n}}{(q^{s+\lambda};q)_{N+n}(q^{s+\lambda};q^\lambda)_{N+d}(q^\lambda;q^\lambda)_{n-d}}$$
- $$D_{n,d} = \mathbf{i}^{\frac{s-r}{2}} q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d-1)}{2}-\lambda Nn+\frac{r-s}{4}-\frac{s(N-d)}{2}} \times \Big(1-q^{\lambda N(N+1)-\lambda d^2-\lambda n+r+s(N-1-d)}\Big)\frac{(q^{s+\lambda};q^\lambda)_{N+n}}{(q^\lambda;q^\lambda)_{N-n}(q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}} \times \sqrt{\frac{1-q^{2\lambda d+s}}{(1-q^{\lambda N(N+1)-\lambda d(d-1)+r+s(N-d)})(1-q^{\lambda N(N+1)-\lambda d(d+1)+r+s(N-1-d)})(1-q^{r-s})}}$$
- $$D^{-1}_{n,d} = \mathbf{i}^{\frac{r-s}{2}} q^{2\lambda d(N-n)-\frac{r-s}{4}+s(N-n)} \times \Big(1-q^{\lambda N(N+1)-\lambda n^2+\lambda d+r+s(N-n)}\Big)\frac{(q^\lambda;q^\lambda)_{N-d}(q^{s+\lambda};q^\lambda)_{n+d-1}}{(q^{s+\lambda};q^\lambda)_{N+d}(q^\lambda;q^\lambda)_{n-d}} \times \sqrt{\frac{(1-q^{2\lambda n+s})(1-q^{r-s})}{(1-q^{\lambda N(N+1)-\lambda n(n-1)+r+s(N-n)})(1-q^{\lambda N(N+1)-\lambda n(n+1)+r+s(N-1-n)})}}$$

5. The matrix $\mathcal{V}_N^{-1}$

We consider $\mathcal{V}_N$ with entries

$$\frac{L_{\lambda(n+d)+r}}{L_{\lambda(n+d)+s}}$$

and again

$$\mathcal{V}_N^{-1} = A\cdot B = D\cdot D^T.$$

Theorem 4. *For $1\le d\le n\le N$:*

- $$A_{n,d} = (-1)^{n-d} q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d+1)}{2}+\lambda Nd-\lambda Nn}\frac{1-(-1)^{N+d}q^{\lambda N(N+1)+sN+r-(d+1)s-\lambda d^2-\lambda n}}{1-(-1)^{N+d}q^{\lambda N(N+1)+sN+r-(d+1)s-\lambda d^2-\lambda d}} \times \frac{(-q^{s+\lambda};q^\lambda)_{N+n}(q^\lambda;q^\lambda)_{N-d}(-q^{s+\lambda};q^\lambda)_{2d}}{(q^\lambda;q^\lambda)_{N-n}(-q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}(-q^{s+\lambda};q^\lambda)_{N+d}}$$
- $$A^{-1}_{n,d} = q^{\lambda n^2-\lambda Nn+\lambda Nd-\lambda nd}\frac{1+(-1)^{N+n}q^{\lambda d+\lambda N(N+1)-\lambda n^2+r+s(N-n)}}{1+(-1)^{N+n}q^{\lambda n+\lambda N(N+1)-\lambda n^2+r+s(N-n)}} \times \frac{(-q^{s+\lambda};q^\lambda)_{n+d-1}(-q^{s+\lambda};q^\lambda)_{N+n}(q^\lambda;q^\lambda)_{N-d}}{(-q^{s+\lambda};q^\lambda)_{2n-1}(q;q)_{n-d}(q^\lambda;q^\lambda)_{N-n}(-q^{s+\lambda};q^\lambda)_{N+d}}$$

- $$B_{d,n}=(-1)^{N+n}\mathbf{i}^{s-r}q^{\lambda\frac{n^2}{2}+\lambda\frac{3d^2}{2}-\lambda Nn-\lambda Nd+\lambda\frac{n-d}{2}+sd-Ns-\frac{s-r}{2}}$$
$$\times\frac{1-(-1)^{N+d}q^{\lambda N(N+1)-\lambda d^2-\lambda n+r+s(N-d-1)}}{1+(-1)^{N+d}q^{\lambda N(N+1)-\lambda d^2+\lambda d+r+s(N-d)}}$$
$$\times\frac{(-q^{s+\lambda};q^\lambda)_{N+n}(-q^{s+\lambda};q^\lambda)_{N+d}}{(-q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{N-n}(-q^{s+\lambda};q^\lambda)_{2d-1}(q^\lambda;q^\lambda)_{N-d}}\frac{1}{1-q^{r-s}}$$
- $$B^{-1}_{d,n}=(-1)^{N+n}\mathbf{i}^{r-s}q^{-\lambda n^2+\lambda Nn+\lambda Nd-\lambda nd-\frac{r-s}{2}+s(N-n)}$$
$$\times\frac{1+(-1)^{N+n}q^{\lambda N(N+1)-\lambda n^2+\lambda d+r+s(N-n)}}{1-(-1)^{N+n}q^{\lambda N(N+1)-\lambda n^2-\lambda n+r+s(N-n-1)}}$$
$$\times\frac{(q^\lambda;q^\lambda)_{N-n}(q^\lambda;q^\lambda)_{N-d}(-q^{s+\lambda};q^\lambda)_{n+d-1}(-q^{s+\lambda};q)_{2n}}{(-q^{s+\lambda};q)_{N+n}(-q^{s+\lambda};q^\lambda)_{N+d}(q^\lambda;q^\lambda)_{n-d}}(1-q^{r-s})$$
- $$D_{n,d}=(-1)^{N+d}\mathbf{i}^{\frac{s-r}{2}}q^{\lambda\frac{n(n+1)}{2}-\lambda\frac{d(d-1)}{2}-\lambda Nn+\frac{r-s}{4}-\frac{s(N-d)}{2}}\sqrt{\frac{1+q^{2\lambda d+s}}{1-q^{r-s}}}$$
$$\times\Big(1-(-1)^{N+d}q^{\lambda N(N+1)-\lambda d^2-\lambda n+r+s(N-1-d)}\Big)\frac{(-q^{s+\lambda};q^\lambda)_{N+n}}{(q^\lambda;q^\lambda)_{N-n}(-q^{s+\lambda};q^\lambda)_{n+d}(q^\lambda;q^\lambda)_{n-d}}$$
$$\times\sqrt{\frac{1}{(1+(-1)^{N+d}q^{\lambda N(N+1)-\lambda d(d-1)+r+s(N-d)})(1-(-1)^{N+d}q^{\lambda N(N+1)-\lambda d(d+1)+r+s(N-1-d)})}}$$
- $$D^{-1}_{n,d}=(-1)^{N+n}\mathbf{i}^{\frac{r-s}{2}}q^{2\lambda d(N-n)-\frac{r-s}{4}+s(N-n)}$$
$$\times\Big(1+(-1)^{N+n}q^{\lambda N(N+1)-\lambda n^2+\lambda d+r+s(N-n)}\Big)\frac{(q^\lambda;q^\lambda)_{N-d}(-q^{s+\lambda};q^\lambda)_{n+d-1}}{(-q^{s+\lambda};q^\lambda)_{N+d}(q^\lambda;q^\lambda)_{n-d}}$$
$$\times\sqrt{\frac{(1+q^{2\lambda n+s})(1-q^{r-s})}{(1+(-1)^{N+n}q^{\lambda N(N+1)-\lambda n(n-1)+r+s(N-n)})(1-(-1)^{N+n}q^{\lambda N(N+1)-\lambda n(n+1)+r+s(N-1-n)})}}$$

6. Some sample proofs

We consider AA^{-1} related to Section 2.

$$\sum_d A_{m,d}A^{-1}_{d,n}=\mathbf{i}^{\lambda mk+2m-\lambda kn}q^{\lambda\frac{m(m+1)}{2}-\lambda Nm-\lambda\frac{km}{2}+\lambda Nn+\lambda\frac{kn}{2}}$$
$$\times\frac{(q^{\lambda+r};q^\lambda)_{N+m+k-1}(q^\lambda;q^\lambda)_{N-n}}{(q^\lambda;q^\lambda)_{N-m}(q^{\lambda+r};q^\lambda)_{N+n+k-1}}$$
$$\times\sum_d(-1)^d q^{\lambda\frac{d(d-1)}{2}-\lambda dn}$$

$$\times \frac{(q^{\lambda+r};q^{\lambda})_{2d}}{(q^{\lambda+r};q)_{m+d}(q^{\lambda};q^{\lambda})_{m-d}} \frac{(q^{\lambda+r};q^{\lambda})_{d-1+n}}{(q^{\lambda+r};q^{\lambda})_{2d-1}(q^{\lambda};q^{\lambda})_{d-n}}.$$

With $q^{\lambda}=Q$ and $q^{r}=b$, the sum in question is

$$\sum_{d}(-1)^{d}Q^{\frac{d(d-1)}{2}-dn}\frac{(bQ;Q)_{2d}}{(bQ;Q)_{m+d}(Q;Q)_{m-d}}\frac{(bQ;Q)_{d-1+n}}{(bQ;Q)_{2d-1}(Q;Q)_{d-n}}.$$

Essentially, this sum has appeared already in an earlier paper [4]. The q-Zeilberger algorithm [8] evaluates it to 0 for $m>n$. For $m=n$, a direct evaluation produces

$$\sum_{d}A_{m,d}A^{-1}_{d,m}=A_{m,m}A^{-1}_{m,m}=1. \quad \square$$

Now we present one proof related to Section 4, namely that

$$\sum_{n\le d\le m}A_{m,d}A^{-1}_{d,n}=\begin{cases}0 & \text{for } n<m,\\ 1 & \text{for } n=m.\end{cases}$$

This is easy to check for $m=n$, so let us assume that $n<m$. Here, Zeilberger's algorithm (as implemented in [8]) does not seem to work. We will present, in order not to overburden the notation, the representative case $\lambda=1$, $r=0$, $s=1$. By performing computer experiments (*creative guessing*) we found that for $n\le K\le m$

$$\sum_{n\le d\le K}A_{m,d}A^{-1}_{d,n}=q^{\frac{K(K+1)}{2}-n(K+1)+Nn-Nm+\frac{m(m+1)}{2}}(-1)^{m-K}\frac{1-q^{N(N+2)+n-m-(K+1)^2}}{1-q^{N(N+2)-(K+1)^2}}$$
$$\times\frac{(q;q)_{N+m+1}(q;q)_{N-n}(q;q)_{n+1+K}}{(q;q)_{m+1+K}(q;q)_{N+1+n}(q;q)_{m-1-K}(q;q)_{K-n}(q;q)_{N-m}(1-q^{m-n})}.$$

We denote temporarily the explicit formula by $\Psi(K)$. Once this result has been established, we have the desired conclusion for $K=m$, since there is a $(q;q)_{-1}$ in the denominator. The formula itself will be proved by induction. For $K=n$ it is easy to check, and otherwise we must show that

$$\Psi(K-1)+A_{m,K}A^{-1}_{K,n}=\Psi(K).$$

After a few cancellations, this amounts to prove that

$$-\frac{1-q^{N(N+2)+n-m-K^2}}{1-q^{N(N+2)-K^2}}\frac{(1-q^{m+K+1})(1-q^{K-n})}{(1-q^{m-n})}$$
$$+(1-q^{2K+1})\frac{1-q^{n+N(N+1)-K^2+N-K}}{1-q^{N(N+1)-K^2+N}}\frac{1-q^{N(N+1)+N-K-1-K^2-m}}{1-q^{N(N+1)+N-2K-1-K^2}}$$
$$=q^{K-n}\frac{1-q^{N(N+2)+n-m-(K+1)^2}}{1-q^{N(N+2)-(K+1)^2}}\frac{(1-q^{n+1+K})(1-q^{m-K})}{(1-q^{m-n})},$$

which is easy to check (best by a computer). Once again, the difficult part here is to guess the correct formula. We hope that the future will bring extensions of Zeilberger's algorithm that do such a proof automatically.

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