Signed β-expansions of minimal weight

Wolfgang Steiner
(joint work with Christiane Frougny)

LIAFA, CNRS, Université Paris 7

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Expansions in base 2

Every integer $N \geq 0$ has an expansion in base 2

$$N = \sum_{j=0}^{K} \epsilon_j 2^j = \epsilon_K \cdots \epsilon_1 \epsilon_0 .$$

with $\epsilon_j \in \{0, 1\}$, which is unique up to leading zeros.
If we allow negative digits, then the number of non-zero digits can often be reduced:

$$7 = 4 + 2 + 1 = 111. = 100\bar{1}. = 8 - 1 \qquad (\bar{1} = -1)$$

Problem: find an expansion of N of minimal weight $\sum_{j=0}^{K} |\epsilon_j|$
(cf. Hamming weight: number of non-zero digits ϵ_j, equal to this weight if $\epsilon_j \in \{-1, 0, 1\}$)

Expansions of minimal weight in base 2

Booth (1951), Reitwiesner (1960), ...: Non-Adjacent Form (NAF)
Every integer N has a unique expansion $N = \sum_{j=0}^{K} \epsilon_j 2^j$ with $\epsilon_j \in \{-1, 0, 1\}$ such that $\epsilon_{j-1} = \epsilon_{j+1} = 0$ if $\epsilon_j \neq 0$. The weight of this expansion is minimal among all expansions of N in base 2.

Dajani, Kraaikamp, Liardet (2006): ergodic properties of the dynamical system associated with the NAF,
$T : [-2/3, 2/3) \to [-2/3, 2/3)$, $T(x) = 2x - \lfloor (3x+1)/2 \rfloor$

Heuberger (2004): $\epsilon_K \cdots \epsilon_1 \epsilon_0 \in \{\bar{1}, 0, 1\}^*$ is a signed 2-expansion of minimal weight if and only if contains none of the factors

$$11(01)^*1,\ 1(0\bar{1})^*\bar{1},\ \bar{1}\bar{1}(0\bar{1})^*\bar{1},\ \bar{1}(01)^*1.$$

(A^* is the free monoid over the set A,
$a^* = \{a\}^* = \{\text{empty word}, a, aa, aaa, aaaa, \dots\}$)

joint digit expansions: Solinas; Grabner, Heuberger, Prodinger

β-expansions, $\beta = \frac{1+\sqrt{5}}{2}$

Greedy β-expansions: Every $x \in \mathbb{R}^+$ has a unique expansion

$$x = \sum_{j \in \mathbb{Z}} \epsilon_j \beta^{-j} = \cdots \epsilon_{-1}\epsilon_0 . \epsilon_1 \epsilon_2 \cdots$$

with $\epsilon_j \in \{0, 1\}$, $\epsilon_{j-1} = \epsilon_{j+1} = 0$ if $\epsilon_j = 1$, which does not terminate with $(10)^\omega = 101010\cdots$.

$$\beta^2 = \beta + 1,\ 100. = 011.,\ 1. = .11$$

Greedy β-expansions are not minimal in weight for $\epsilon_j \in \{-1, 0, 1\}$:

$$0101001. = 10\bar{1}1001. = 1000\bar{1}01. = 10000\bar{1}0.$$

β-expansions of minimal weight

$x = x_1 \cdots x_n \in A_\beta^*$ is β-heavy if it is not minimal in weight, i.e., if there exists $y = y_\ell \cdots y_r \in A_\beta^*$ with

$$\mathbf{.}x_1 \cdots x_n = y_\ell \cdots y_0 \mathbf{.} y_1 \cdots y_r \quad \text{and} \quad \sum_{j=\ell}^{r} |y_j| < \sum_{j=1}^{n} |x_j|.$$

If $x_1 \cdots x_{n-1}$ and $x_2 \cdots x_n$ are not β-heavy, x is strictly β-heavy.

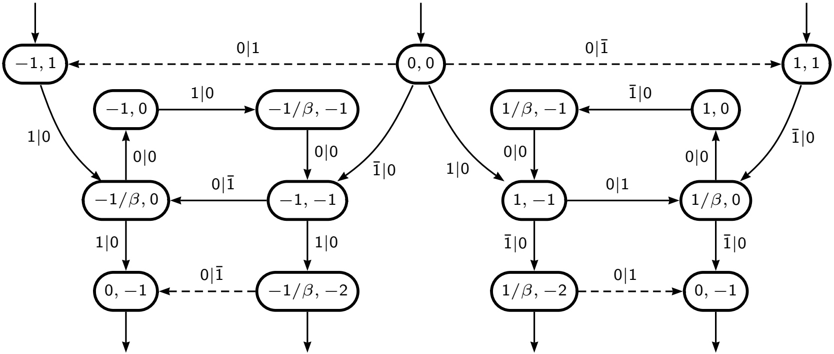
Theorem

If $\beta = \frac{1+\sqrt{5}}{2}$, then the set of strictly β-heavy words is

$$1(0100)^*1 \;\cup\; 1(0100)^*0101 \;\cup\; 1(00\bar{1}0)^*\bar{1} \;\cup\; 1(00\bar{1}0)^*0\bar{1}$$
$$\cup\; \bar{1}(0\bar{1}00)^*\bar{1} \;\cup\; \bar{1}(0\bar{1}00)^*0\bar{1}0\bar{1} \;\cup\; \bar{1}(0010)^*1 \;\cup\; \bar{1}(0010)^*01.$$

If $\cdots \epsilon_{-1}\epsilon_0\epsilon_1 \cdots$ does not contain any of these factors, then $\cdots \epsilon_{-1}\epsilon_0 \mathbf{.} \epsilon_1 \cdots$ is a signed β-expansion of minimal weight.

The strictly β-heavy words are the inputs of the following transducer. The outputs are corresponding lighter words (if the path is completed by dashed arrows such that it runs from $(0,0)$ to $(0,-1)$).



$$(s,\delta) \xrightarrow{a|b} (s',\delta') : \; s' = \beta s + a - b, \; \delta' = \delta + |b| - |a|$$

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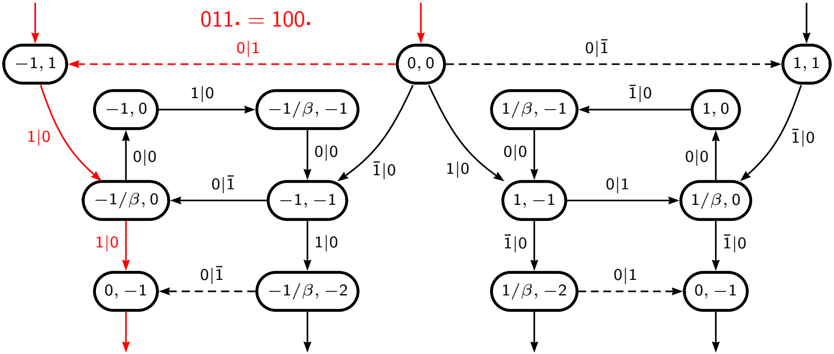

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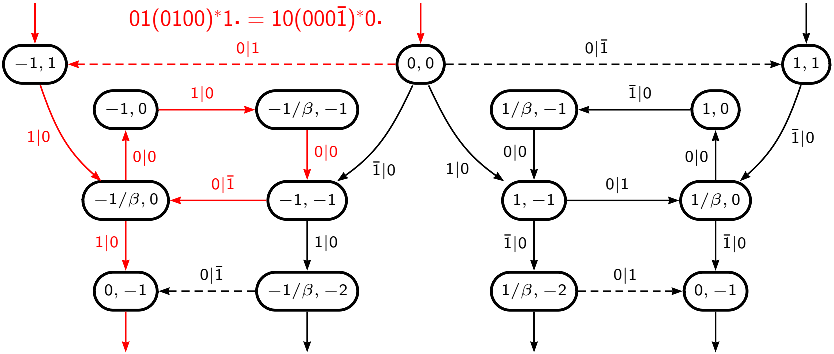
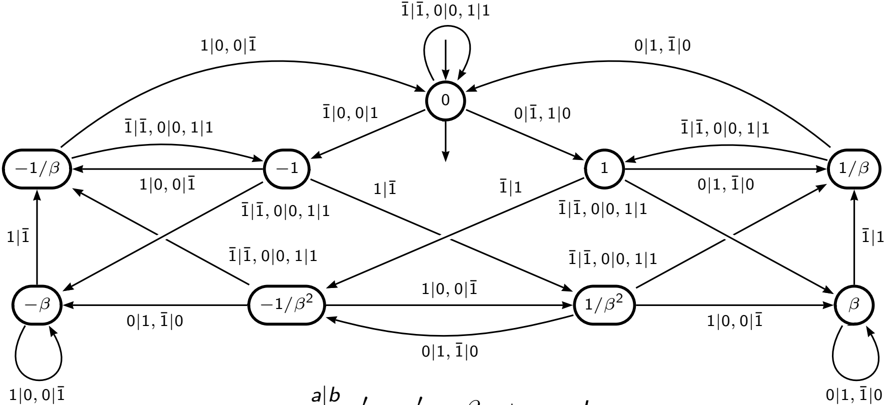


$$(s,\delta) \xrightarrow{a|b} (s',\delta') : \; s' = \beta s + a - b, \; \delta' = \delta + |b| - |a|$$

The sequences $x_1|y_1, \ldots, x_n|y_n$ with $x_1 \cdots x_n, y_1 \cdots y_n \in \{\bar{1}, 0, 1\}^*$ (not containing a factor 11 or $\bar{1}\bar{1}$) such that $\centerdot x_1 \cdots x_n = \centerdot y_1 \cdots y_n$ are recognized by the redundancy automaton (transducer)

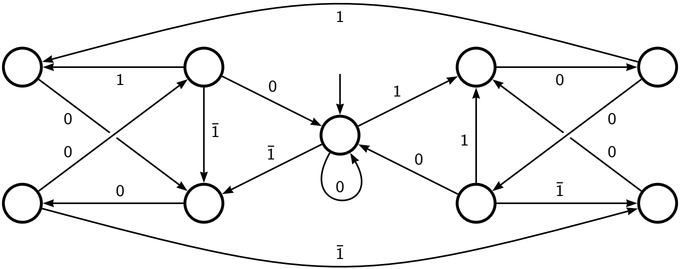


$$s \xrightarrow{a|b} s' : \; s' = \beta s + a - b$$

If $s_0 = 0$, $s_{j-1} \xrightarrow{x_j|y_j} s_j$, $1 \le j \le n$, then $s_j = x_1 \cdots x_j \centerdot - y_1 \cdots y_j \centerdot$, and $\centerdot x_1 \cdots x_n = \centerdot y_1 \cdots y_n$ if and only if $s_n = 0$.

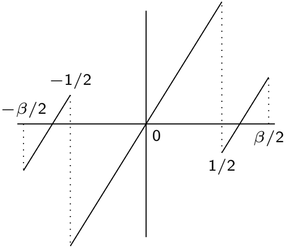
Theorem

For $\beta = \frac{1+\sqrt{5}}{2}$, *the signed* β*-expansions of minimal weight are given by the following automaton, where all states are terminal.*



Transformation providing signed β-expansions of minimal weight, $\beta = \frac{1+\sqrt{5}}{2}$

$$T : [-\beta/2, \beta/2) \to [-\beta/2, \beta/2),\ T(x) = \beta x - \lfloor x + 1/2 \rfloor$$



Proposition

If $x \in [\beta/2, \beta/2)$ *and* $x_j = \lfloor T^{j-1}(x) + 1/2 \rfloor$, *then* $x = .x_1 x_2 \cdots$ *is a signed* β*-expansion of minimal weight avoiding the factors* 11, 101, $1\bar{1}$, $10\bar{1}$, $100\bar{1}$ *and their opposites.*

Proof of the proposition.

Recall that $T(x) = \beta x - \lfloor x + 1/2 \rfloor$ and $x_j = \lfloor T^{j-1}(x) + 1/2 \rfloor$. If $x_j = 1$, then $T^{j-1}(x) \in [1/2, \beta/2) = [\boldsymbol{.}(010)^\omega, \boldsymbol{.}(100)^\omega)$,

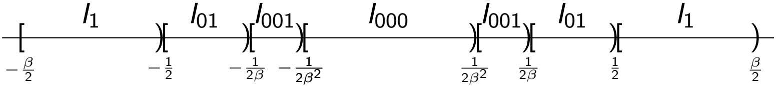
$$\begin{aligned} T^j(x) &\in [\beta/2 - 1, \beta^2/2 - 1) = [-1/(2\beta^2), 1/(2\beta)), \ x_{j+1} = 0, \\ T^{j+1}(x) &\in [-1/(2\beta), 1/2), \qquad x_{j+2} = 0, \\ T^{j+2}(x) &\in [-1/2, \beta/2), \qquad x_{j+3} \in \{0, 1\}. \end{aligned}$$

Hence 11, 101, $1\bar{1}$, $10\bar{1}$ and $100\bar{1}$ are avoided, thus the strictly β-heavy words $1(0100)^*1$, $1(0100)^*0101$, $1(00\bar{1}0)^*\bar{1}$, $1(00\bar{1}0)^*0\bar{1}$ are avoided. The same is true for the opposite words. □

Remark. Heuberger (2004) excluded (for the Fibonacci numeration system) the factor 1001 instead of $100\bar{1}$. This can be achieved by $T(x) = \beta x - \left\lfloor \frac{\beta^2+1}{2\beta} x + \frac{1}{2} \right\rfloor$ on $\left[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\right)$, $\frac{\beta^2}{\beta^2+1} = \boldsymbol{.}(1000)^\omega$.

Markov chain of digits

Let $T(x) = \beta x - \lfloor x + 1/2 \rfloor$, and $I_{000}, I_{001}, I_{01}, I_1$ as follows



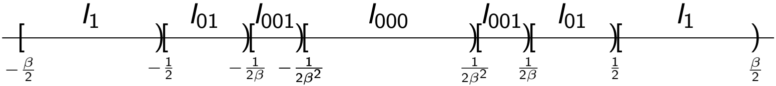
The sequence of random variables $(X_k)_{k\geq 0}$ defined by

$$\begin{aligned}
&\Pr[X_0 = j_0, \ldots, X_k = j_k] \\
&= \lambda(\{x \in [-\beta/2, \beta/2) : x \in I_{j_0}, T(x) \in I_{j_1}, \ldots, T^k(x) \in I_{j_k}\})/\beta \\
&\qquad = \lambda(I_{j_0} \cap T^{-1}(I_{j_1}) \cap \cdots \cap T^{-k}(I_{j_k}))/\beta
\end{aligned}$$

(where λ denotes the Lebesgue measure) is a Markov chain since

$$T(I_{000}) = I_{000} \cup I_{001} = T(I_1), \;\; T(I_{001}) = I_{01}, \;\; T(I_{01}) = I_1$$

and $T(x)$ is linear on each I_j.



The matrix of transition probabilites is

$$(\Pr[X_k = j \mid X_{k-1} = i])_{i,j\in\{000,001,01,1\}} = \begin{pmatrix} 1/\beta & 1/\beta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2/\beta^2 & 1/\beta^3 & 0 & 0 \end{pmatrix}$$

the stationary distribution vector is $(2/5, 1/5, 1/5, 1/5)$. Therefore

$$\Pr[X_k = 1] = \lambda(\{x \in [-\beta/2, \beta/2) : T^k(x) \in I_1\} \to 1/5,$$

i.e., the expected number of non-zero digits in a signed β-expansion of minimal weight of length n is $n/5 + \mathcal{O}(1)$.
(cf. greedy β-expansions $n/(\beta^2 + 1)$, base 2 minimal expansions $n/3$)

Fibonacci numeration system

Let $F_0 = 1$, $F_1 = 2$, $F_j = F_{j-1} + F_{j-2}$. Then every integer $N \geq 0$ has a unique F-expansion

$$N = \sum_{j=1}^{n} \epsilon_j F_{n-j} = \langle \epsilon_1 \cdots \epsilon_n \rangle_F$$

with $\epsilon_j \in \{0, 1\}$ and $\epsilon_{j-1} = \epsilon_{j+1} = 0$ if $\epsilon_j = 1$.

$x_1 \cdots x_n \in \{\bar{1}, 0, 1\}^*$ is F-heavy if there exists $y_\ell \cdots y_n \in \{\bar{1}, 0, 1\}^*$ such that $\langle x_1 \cdots x_n \rangle_F = \langle y_\ell \cdots y_n \rangle_F$ and $\sum_{j=\ell}^{n} |y_j| < \sum_{j=1}^{n} |x_j|$.

$\langle \cdots 1\bar{1}0 \cdots \rangle_F = \langle \cdots 001 \cdots \rangle_F$, but $\langle \cdots 1\bar{1} \rangle_F = \langle \cdots 01 \rangle_F$.

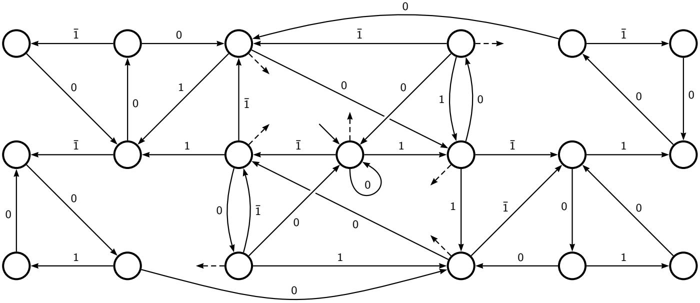
Theorem

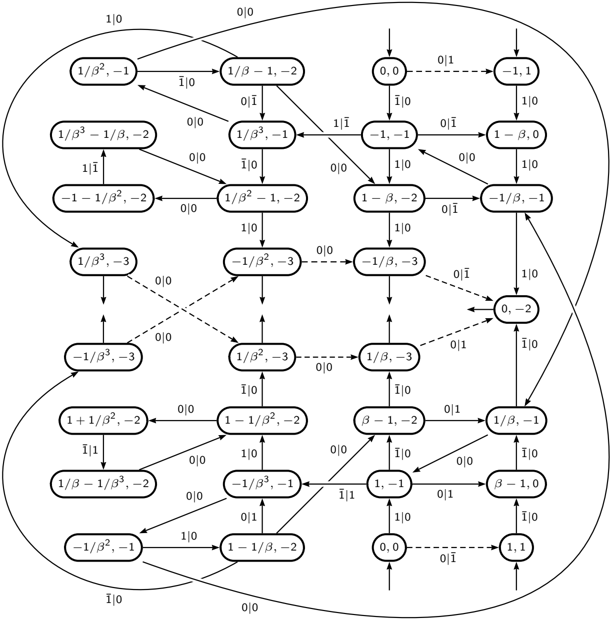
The F-heavy words are exactly the β-heavy words for $\beta = \frac{1+\sqrt{5}}{2}$, i.e. a word is a signed F-expansion of minimal weight if and only if it is a signed β-expansion of minimal weight.

Tribonacci numeration system

If $\beta > 1$ is the Tribonacci number, $\beta^3 = \beta^2 + \beta + 1$, then the strictly β-heavy words are the inputs of the automaton on the next slide. The signed β-expansions of minimal weight are given by the following automaton where all states are terminal.

If $T_0 = 1,\ T_1 = 2,\ T_2 = 4,\ T_j = T_{j-1} + T_{j-2} + T_{j-3}$, then the signed T-expansions of minimal weight are given by the automaton where only the states with dashed outgoing arrows are terminal.





1|0
0|0
$1/\beta^2, -1$
$\bar{1}|0$
$1/\beta - 1, -2$
$0, 0$
0|1
$-1, 1$
$0|\bar{1}$
$\bar{1}|0$
1|0
$1/\beta^3 - 1/\beta, -2$
0|0
$1/\beta^3, -1$
$1|\bar{1}$
$-1, -1$
$0|\bar{1}$
$1 - \beta, 0$
$1|\bar{1}$
0|0
$\bar{1}|0$
0|0
1|0
0|0
1|0
$-1 - 1/\beta^2, -2$
0|0
$1/\beta^2 - 1, -2$
$1 - \beta, -2$
$0|\bar{1}$
$-1/\beta, -1$
1|0
1|0
$1/\beta^3, -3$
0|0
$-1/\beta^2, -3$
0|0
$-1/\beta, -3$
$0|\bar{1}$
1|0
$0, -2$
0|0
$-1/\beta^3, -3$
$1/\beta^2, -3$
0|0
$1/\beta, -3$
0|1
$\bar{1}|0$
$\bar{1}|0$
$\bar{1}|0$
$1 + 1/\beta^2, -2$
0|0
$1 - 1/\beta^2, -2$
$\beta - 1, -2$
0|1
$1/\beta, -1$
$\bar{1}|1$
0|0
1|0
0|0
$\bar{1}|0$
0|0
$\bar{1}|0$
$1/\beta - 1/\beta^3, -2$
$-1/\beta^3, -1$
$\bar{1}|1$
$1, -1$
0|1
$\beta - 1, 0$
0|0
0|1
1|0
$\bar{1}|0$
$-1/\beta^2, -1$
1|0
$1 - 1/\beta, -2$
$0, 0$
$0|\bar{1}$
$1, 1$
$\bar{1}|0$
0|0

Particular signed β-expansions of minimal weight, $\beta^3 = \beta^2 + \beta + 1$

$$T : \left[\tfrac{-\beta^2}{\beta^2+1}, \tfrac{\beta^2}{\beta^2+1}\right) \to \left[\tfrac{-\beta^2}{\beta^2+1}, \tfrac{\beta^2}{\beta^2+1}\right),\ T(x) = \beta x - \left\lfloor \tfrac{\beta^2+1}{2\beta} x + \tfrac{1}{2} \right\rfloor$$

Proposition

If $x \in \left[\frac{-\beta^2}{\beta^2+1}, \frac{\beta^2}{\beta^2+1}\right)$ and $x_j = \left\lfloor \frac{\beta^2+1}{2\beta} T^{j-1}(x) + 1/2 \right\rfloor$, then $x = .x_1 x_2 \cdots$ is a signed β-expansion of minimal weight avoiding the factors 11, $1\bar{1}$, $10\bar{1}$ and their opposites.

The transition probabilities of the corresponding Markov chain are

$$\begin{pmatrix} 1/\beta & 1-1/\beta & 0 \\ 0 & 0 & 1 \\ 1-1/\beta^2 & 1/\beta^2 & 0 \end{pmatrix}$$

the stationary distribution vector is $\left(\frac{\beta^3+\beta^2}{\beta^5+1}, \frac{\beta^3}{\beta^5+1}, \frac{\beta^3}{\beta^5+1}\right)$, hence the probability of a non-zero digit is asymptotically $\frac{\beta^3}{\beta^5+1} \approx 0.28219$.

General case, condition (D)

(D): $\beta > 1$ and $P(\beta) = 0$ for some polynomial
$P(X) = X^d - b_1 X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$ with $b_1 > \sum_{j=2}^{d} |b_j|$

If β satisfies (D), then β is a Pisot number.

Proposition (Akiyama, Rao, St. (2004))

Let β satisfy (D), and $x_1 \cdots x_n \in \mathbb{Z}^*$ such that $|.x_1 \cdots x_n| < 1$. Then there exists a word $y_0 \cdots y_m \in \{-\lfloor\beta\rfloor, \ldots, \lfloor\beta\rfloor\}^*$ such that $y_0.y_1 \cdots y_m = .x_1 \cdots x_n$ and $\sum_{j=0}^{m} |y_j| \leq \sum_{j=1}^{n} |x_j|$.

(More precisely, the positive and negative parts of $y_0 \cdots y_m$ can be chosen to be β-admissible, hence β satisfies (W).)

Theorem

If β satisfies (D), then the set of signed β-expansions of minimal weight is recognized by a finite automaton, which is computable.