Topological properties of central tiles for substitutions

Joerg Thuswaldner, Anne Siegel

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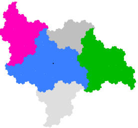
Central Tiles and Rauzy fractals

Introduced by Rauzy and Thurston in different frameworks

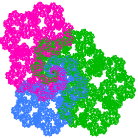
- Symbolic dynamical systems Geometrical representation of the shift map on a substitutive dynamical system. The shift map commutes with a piecewise exchange of domains.
- Beta-numeration Geometric compact representation of real numbers with an empty fractional greedy expansion in a non-integer numeration system.
- Discrete geometry Renormalized limit of an inflation action on faces of discrete planes.

Specific topological properties

0 inner point

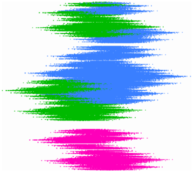


(0 inner point)



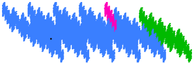
(0 not inner point)

connectivity

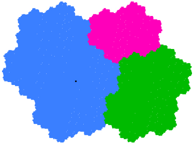


(not connected)

Haussdorf dimension of the boundary



disklikeness Parametrization of the boundary



Give criterions for topological properties that can be checked algorithmically ?

Definitions

- Substitution. endomorphism σ of the free monoid $\{0, \ldots, n\}^*$.
  $\sigma: \quad 1 \to 12 \quad 2 \to 13 \quad 3 \to 1. \qquad (\beta^3 = \beta^2 + \beta + 1)$
- Primitivity. The map $\mathbf{M}$ obtained by abelianization of $0, \ldots, n^*$ on σ is primitive.
- Periodic points. If σ is primitive, then there exists at least a periodic point w for σ:
  $$\sigma^\nu(w) = w.$$
- unit Pisot assumption The dominant eigenvalue β of the abelianized matrix of σ is a unit Pisot number.
  $\sigma: \quad 1 \to 12 \quad 2 \to 3 \quad 3 \to 1 \quad 4 \to 5 \quad 5 \to 1 \qquad (\beta^3 = \beta + 1)$
  Let $d \leq n$ be the algebraic degree of β. Let Min_β be its minimal polynomial.

Central Tile

- Beta-decomposition of the space:
  - Beta-expanding line $\mathbb{H}_e$
  - Beta-contracting space $\mathbb{H}_c$ generated by the eigenvectors for the algebraic conjugates β_i's of β.
  - Beta-Orthogonal space: subspace $\mathbb{H}_o$ generated by the other eigenvectors.
- Beta-projection: projection on the beta-contracting plane parrallel to $GH_e + \mathbb{H}_o$

$$\forall w \in \mathcal{A}^*, \; \pi(\mathbf{l}(\sigma(w))) = \mathbf{h}\pi(\mathbf{l}(w)).$$

Central Tile

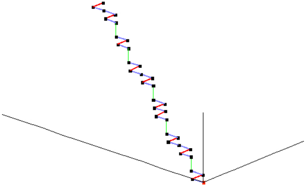
$\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 4$, $\sigma(4) = 1$

112 112 113 112 112 113 112 112 4 112 112 113 112 112 113 112 112 113 112 112 4 112 112 113 112 112 113 1 112 ...

Construction of the central tile

- Compute a periodic point
- Embed it as a stair in $\mathbb{R}^n$.
- Project the stair on the beta-contracting plane
- Keep memory of the type of step when projecting
- Take the closure

Central Tile



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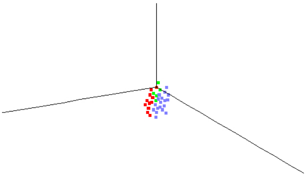

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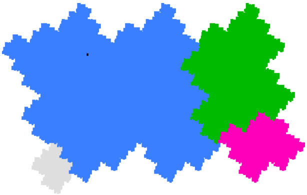

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- Take the closure

Definition

Let σ be a primitive unit Pisot substitution. The central tile of σ is defined by

$$\mathcal{T}_\sigma = \overline{\{\pi(\mathbf{l}(u_0 \cdots u_{i-1}));\ i \in \mathbb{N}\}}.$$

$$\text{Subtile: } \mathcal{T}(a) = \overline{\{\pi\left(\mathbf{l}(u_0 \cdots u_{i-1})\right);\ i \in \mathbb{N},\ u_i = a\}}.$$

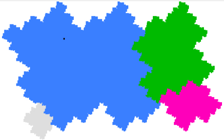
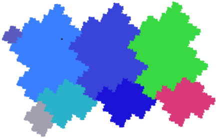
Main topological properties

Theorem

Let σ be a primitive Pisot unit substitution.

- *The central tile $\mathcal{T}$ is a compact subset of $\mathbb{R}^{d-1}$, with nonempty interior and non-zero measure. (d degree of Min_β).*
- *Each subtile is the closure of its interior.*
- *The subtiles of $\mathcal{T}$ are solutions of the following affine Graph Iterated Function System:*
$$\mathcal{T}(a) = \bigcup_{b\in\mathcal{A},\, \sigma(b)=pas} \mathbf{h}(\mathcal{T}(b)) + \pi(\mathbf{l}(p))$$
- *The subtiles are disjoint when the substitution satisfies the so-called coincidence condition.*

$$\mathcal{T}(1) = \mathbf{h}[\mathcal{T}(1) \cup (\mathcal{T}(1) + \pi\mathbf{l}(e_1)) \cup \mathcal{T}(2) \cup (\mathcal{T}(2) + \pi\mathbf{l}(e_1)) \cup \mathcal{T}(4)],$$
$$\mathcal{T}(2) = \mathbf{h}(\mathcal{T}(1) + 2\pi\mathbf{l}(e_1)),$$
$$\mathcal{T}(3) = \mathbf{h}(\mathcal{T}(2) + 2\pi\mathbf{l}(e_1)),$$
$$\mathcal{T}(4) = \mathbf{h}(\mathcal{T}(3)$$

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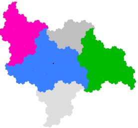
Specific topological properties

O inner point

(Sufficient conditions, CNS conditions) [Rauzy, Akiyama]



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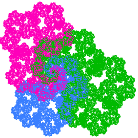
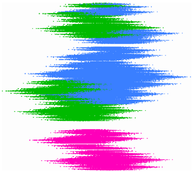


(0 not inner point)

connectivity

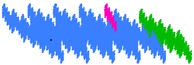
(Sufficient condition, necessary condition) [Canterini, Messaoudi]



(not connected)

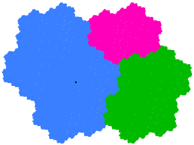
Haussdorf dimension of the boundary

(Examples of computation) [Feng-Furukado-Ito, Thuswaldner]



disklikeness
Parametrization of the boundary

(Examples) [Messaoudi,Sirvent]



Give criterions for topological properties that can be checked

The main object: tilings

A multiple tiling is given by a *translation set* $\Gamma \subset \mathbb{H}_c \times \mathcal{A}$ such that

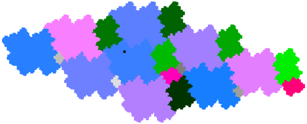
- $\mathbb{H}_c = \bigcup_{(\gamma,i)\in\Gamma} \mathcal{T}_i + \gamma$
- Delaunay set (finite number of intersections for a given tile).
- almost all points in $\mathbb{H}_c$ are covered exactly p times.

Self-replicating substitution multiple tiling

$$\Gamma_{srs} = \{(\pi(\mathbf{x}), i) \in \pi(\mathbb{Z}^n) \times \mathcal{A},\\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

Delaunay set, self-replicating, aperiodic and repetitive.

Tiling iff super-coincidence.



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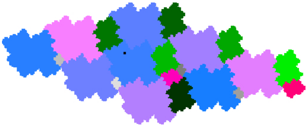
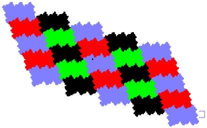


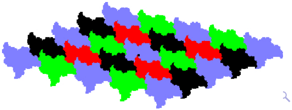
Lattice multiple tiling

$(\mathbf{e}_{\mathbf{B(1)}}, \ldots, \mathbf{e}_{B(d)})$ $\mathbb{Z}$-basis of $\pi(\mathbb{Z}^n)$

$$\Gamma_{lattice} = \{(\pi(\mathbf{x}), i) \in \pi(\mathbb{Z}^n) \times \mathcal{A},\ \sum_1^d \langle \mathbf{x}, \mathbf{e}_{B(k)}\rangle = 0\}.$$

Periodic and Delaunay set.
When σ irreducible, tiling iff super-coincidence.





The main tool: IFS description of intersection of tiles

Suppose that two tiles intersect. $\mathcal{I} = \mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b)) \neq \emptyset$.

Each tile admits a decomposition, hence

$$\mathcal{T}(a) = \bigcup_{\sigma(a_1)=p_1 a s_1} \mathbf{h}(\mathcal{T}(a_1)+\pi\mathbf{l}(p_1)). \qquad \mathcal{T}(b) = \bigcup_{\sigma(b_1)=p_2 b s_2} \mathbf{h}(\mathcal{T}(b_1)+\pi\mathbf{l}(p_2)).$$

Then the union can be rewritten as

$$\begin{aligned} \mathcal{I} &= \bigcup_{\substack{\sigma(a_1)=p_1 a s_1 \\ \sigma(b_1)=p_2 b s_2}} \mathbf{h}[\mathcal{T}(a_1)+\pi\mathbf{l}(p_1)] \cap \{\mathbf{h}[\mathcal{T}(b_1)+\pi\mathbf{l}(p_2)] + \pi(\mathbf{x})\}. \\ &= \bigcup \mathbf{h}\pi\mathbf{l}(p_1) + \mathbf{h}[\mathcal{T}(a_1) \cap (\mathcal{T}(b_1) + \pi\mathbf{l}(p_2) - \pi\mathbf{l}(p_1) + \mathbf{h}^{-1}\pi(\mathbf{x}))] \end{aligned}$$

The boundary graph maps the intersection of two tiles to each intersections that is contained in it (up to a translation).

$$(\mathbf{0}, a) \cap (\pi(\mathbf{x}), b) \to (\mathbf{0}, a_1) \cap (\pi\mathbf{l}(p_2) - \pi\mathbf{l}(p_1) + \mathbf{h}^{-1}\pi(\mathbf{x}), b_1)$$

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Self-replicating substitution neighbor graph

- Nodes: pairs of faces $[(\mathbf{0}, a), (\pi(\mathbf{x}), b)]$ such that
  - $(\pi(\mathbf{x}), b) \in \Gamma srs$ (points in the translation set)
  - $||\pi(\mathbf{x})|| \leq ||\mathcal{T}||$ (if not, the intersection is empty)
- There is an edge between $(\mathbf{0}, a) \cap (\pi(\mathbf{x}), b)$ and $(\mathbf{0}, a_1) \cap (\pi(\mathbf{x}_1), b_1)$ if $\mathcal{T}(a_1) \cap (\pi(\mathbf{x}) + \mathcal{T}(b_1))$ appears up to a translation in the decomposition of $\mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b))$.

Theorem

The self-replicating substitution boundary graph is finite.

$\mathcal{T}(a) \cap (\pi(\mathbf{x}) + \mathcal{T}(b))$ is nonempty iff the self-replicating substitution boundary graph contains an infinite walk starting in $[(\mathbf{0}, a), (\pi(\mathbf{x}), b)]$.

Each path of the graph correspond to a point lying at the intersection. The boundary graph is a GIFS description of the boundary of the central tile.

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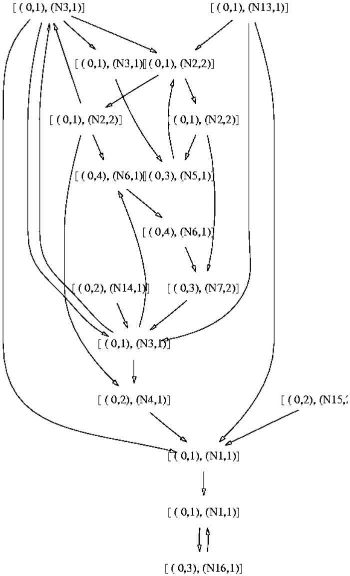
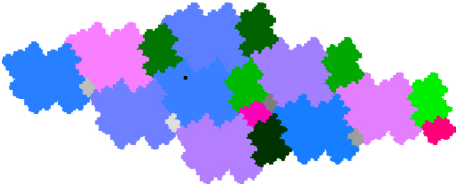
Example





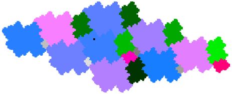
The central subtiles intersect 17 other tiles in the SRS tiling

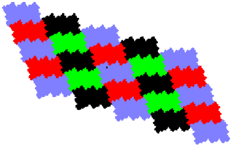
$\mathcal{T}(1)$ has 5 neighbourgs outside the central tile.

Several graphs

It is algorithmically possible to compute graphs

- Self-replicating substitution neighbor graph Pairs of tiles intersecting in the SRS multiple tiling.
- Connectivity graph Pairs of subtiles of $\mathcal{T}(a)$ with a common point.
- Lattice neighbor graph Pairs of tiles in lattice multiple tiling.
- Triple point neighbor graph Triplets of tiles intersecting in the SRS multiple tiling.
- Quadruple point neighbor graph Quadruplets of tiles intersecting in the SRS mutiple tiling.





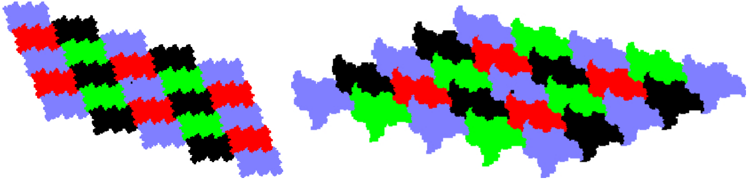
- 6 intersecting pairs in the lattice tiling
- 20 intersecting triplets in the SRS tiling (redundancy)
- 4 intersecting quadruplets in the SRS tiling

Application to boundary

Proposition

The SRS multiple tiling is a tiling iff the dominant eigenvalue of the matrix of the SRS neighbor graph is strictly less than β.
The lattice multiple tiling is a tiling iff the dominant eigenvalue of the matrix of the lattice neighbor graph is strictly less than β.

Application: $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 4$, $\sigma(4) = 1$ generates a lattice tiling.
$\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 4$, $\sigma(4) = 5$, $\sigma(5) = 1$ does not generate a lattice tiling with the given vectors.



Application to boundary

Proposition

Let λ be the largest conjugate of β and λ' the smallest conjugate. Let μ be the dominant eigenvalue of the matrix of the SRS neighbor graph. If the SRS neighor graph is strongly connected then

$$\dim_B(\partial\mathcal{T}) = \dim_B(\partial\mathcal{T}(a)) = d - 1 + \frac{\log\lambda - \log\mu}{\log\lambda'}$$

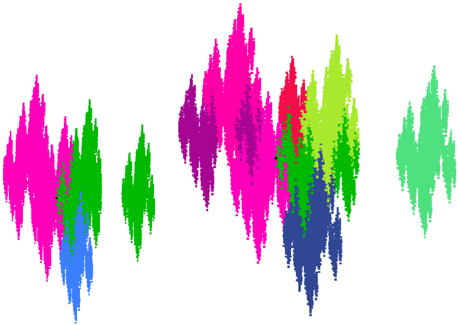
Application: Explicit computations of Haussdorf dimensions.

Application to connectivity

Connectivity graph For each subtile $\mathcal{T}(a)$, their is an edge between two subunits iff they intersect.

Proposition

Each $\mathcal{T}(a)$ is a locally connected continuum if and only if the connectivity graph $G_a(V,E)$ is connected for each $a \in \mathcal{A}$.
$\mathcal{T}$ is connected iff each $\mathcal{T}(a)$ and the subtiles have connections.



$\sigma(1) = 3$; $\sigma(2) = 23$, $\sigma(3) = 31223$.

- The three central tiles intersect.
- One subtile of $\mathcal{T}(2)$ intersects no other subtile: some nodes are missing in the graph.

Criterion for non disklike

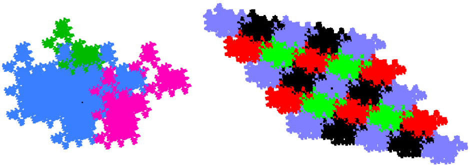
Proposition

Suppose that β has degree 3. If the central tile $\mathcal{T}$ is homeomorphic to a closed disk then $\mathcal{T}$ has at most six neighbors λ in a lattice tiling with the property

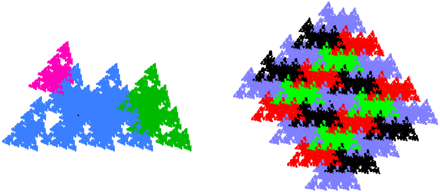
$$|\mathcal{T}_\sigma \cap (\mathcal{T}_\sigma + \gamma)| > 1.$$

Deduced from Bandt and Gelbrich.

Application: when there is lattice tiling, check if the central tile is not disklike.



8 neighbours. Not homeomorphic to a closed disk.



Only 6 neighbors. No conclusion

Criterion for disklike

Theorem

Suppose that β has degree 3. Let $B_1, \ldots B_k$ be the boundary pieces $\mathcal{T}(a) \cap (\mathcal{T}(b) + \pi(\mathbf{x}))$. Suppose that

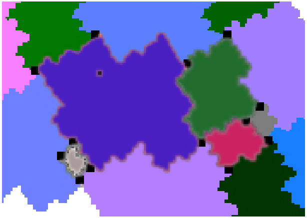
- *The B_i's form a circular chain: they can be arranged so that they have one interesection point with the following and no intersection with the others.*
- *The self-affine decomposition of each B_i is a regular chain*

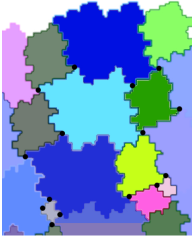
Then the central tile is disklike

Translation into the boundary graph framework. A boundary piece B_i corresponds to a node $[(\mathbf{0}, a), (\pi\mathbf{x}, b)]$.

Algorithmic criterion for disklike

- Identify pairs intersecting as a singleton
- Check that every triple intersection in a singleton.
- For every pair-intersection $[(\mathbf{0}, a), (\pi\mathbf{x}, b)]$ that is not a singleton, check that it intersect exactly two other intersections.
- The intersections make a loop.
- Similar checking for the successors of $[(\mathbf{0}, a), (\pi\mathbf{x}, b)]$.





$$\sigma(1) = 112,\ \sigma(2) = 113,\ \sigma(3) = 4,\ \sigma(4) = 1$$

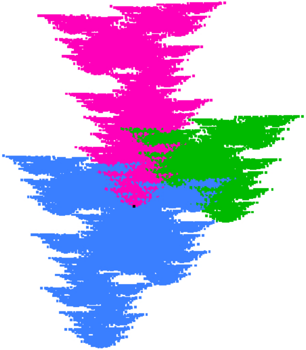
- 17 pair-intersections of tiles.
- 4 contains exactly one point (Sommets 1,15, 16, 17)
- 13 remaining infinite pair-intersections.

The central tile for $\sigma(1) = 112$, $\sigma(2) = 113$, $\sigma(3) = 4$, $\sigma(4) = 1$ is homeomorphic to a closed disk.

Criterion for not simply connected

Theorem

The SRS boundary graph, triple point graph and quadruple point graph allow to check a condition for not simply connected.



Not simply connected

Conclusion

- Many topological properties of central tiles can be checked.
- Understand the structure of boundary, triple and quadruple graphs for classes of substitutions to deduce general properties?
- What is the relation between topological properties and ergodic properties of the substitutive dynamical system?
- What can be deduced from topological properties about beta-numeration systems?
- (Find a good programmer to compute efficiently the graphs to check the conditions?)