STURMIAN SUBSTITUTIONS, CUTTING PATHS AND THEIR PROJECTIONS

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Words

An *alphabet* $\mathcal{A}$ is a finite set of elements that are called *letters*. We take $\mathcal{A} = \{0, 1\}$.

A *word* is a function u from a finite or infinite block of integers to $\mathcal{A}$. If this block of integers contains negative numbers we call u a *central word*. If a word u is finite, we denote by $|u|$ the number of letters in u, and by $|u|_a$ the number of occurrences of the letter a in u.

A word u is called *balanced* if $||v|_0 - |w|_0| < 2$ for all subwords v, w of equal length. A finite word u is called *strongly balanced* if u^2 is balanced. Here u^2 is the concatenation of u with u.

A strongly balanced word is called a *Christoffel word* when it is smaller than each of its shifts in the lexicographic order (the zeros are placed as far to the left as possible).

Central progressions w_u

Definition. Let $u = u(0)\dots u(m-1)$ be a strongly balanced finite word, containing both zeros and ones, with $\gcd(|u|_0, |u|_1) = 1$. The *cutting path* in the x-y-plane corresponding to u consists of $m+1$ integer points p_i given by $p_i = (|u(0)\dots u(i-1)|_0, |u(0)\dots u(i-1)|_1)$ for $i = 0, \dots, m$, connected by line segments of lengths 1.

Draw the line through the origin and the end point of the path, given by $y = \frac{|u|_1}{|u|_0}x$. We project each integer point p_i on the cutting path parallel to this line onto the y-axis. By $P(p_i)$ we denote the second coordinate of the projection of p_i. It is clear that $P(p_0) = P(p_m) = 0$.

Definition. We define the function w_u as follows. If $P(p_i) = k/|u|_0$ then $w_u(-k) = i$. We say w_u has the number i at position $-k$. We call w_u the *central progression* corresponding to u.

Some properties of a central progression w_u corresponding to $u = u(0) \ldots u(m-1)$.

- Its domain is a block of integers of length m of $\mathbb{Z}$ containing 0.

- Its image is the set $\{0, 1, \ldots m-1\}$.

- There exists a $c \in \mathbb{Z}$ such that if k is in the domain of w, then $w(k) \equiv ck \pmod{m}$.

Example 1. Let $u = 01001$, then the central progression w_u is given by $w = 20314$.

Central words v_w

Definition. Let w be a central progression. Then the central word v_w is the word that you get by replacing every number in w that is smaller than its right neighbour by 0, and every number that is larger by 1.

Example 2. If $u = 01001$, then $w_u = 20314$, and $v_w = 1\underline{0}101$, where we underlined the letter at position 0.
If $u = 0110101101101$, then
$w_u = 11\ 3\ 8\ 0\ 5\ 10\ 2\ 7\ 12\ 4\ 9\ 1\ 6$, and
$v_w = 101\underline{0}010010100$.

Sturmian substitutions

A *substitution* σ is an application from the alphabet $\mathcal{A} = \{0, 1\}$ to the set of finite words. It extends to a morphism by concatenation, that is, $\sigma(uv) = \sigma(u)\sigma(v)$.

A *fixed point* of a substitution σ is an infinite word u with $\sigma(u) = u$.

If σ is a substitution, we call

$$M_\sigma = \begin{pmatrix} |\sigma(0)|_0 & |\sigma(0)|_1 \\ |\sigma(1)|_0 & |\sigma(1)|_1 \end{pmatrix}$$

its *incidence matrix*.

A one-sided infinite word is *Sturmian* if it is balanced and not ultimately periodic.

We call a substitution σ over two letters *Sturmian* if $\sigma(u)$ is a Sturmian word for every Sturmian word u.

Let σ be a Sturmian substitution that has incidence matrix with determiant 1 and a fixed point starting with 0, let $u_n = \sigma^n(0)$, let $w_n = w_{u_n}$ and let $v_n = v_{w_n}$ for $n > 0$.

Example 3. Let σ be the substitution defined by $\sigma(0) = 010$, $\sigma(1) = 01$. Then σ is a Sturmian substitution with $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We have

$$\begin{aligned} &u_0 = 0 \\ &u_1 = 010 \\ &u_2 = 01001010 \\ &u_3 = 010010100100101001010 \\ &\cdots\cdots \end{aligned}$$

This yields the following table of central progressions w_n.

n	w_n												
0									0				
1								2	0	1			
2						7	2	5	0	3	6	1	4
3	20	7	15	2	10	18	5	13	0	8	16	3	$\longrightarrow$
	$\longrightarrow$	11	19	6	14	1	9	17	4	12			

....

We get the following table of central words.

n	v_n												
0									$\underline{0}$				
1								1	$\underline{0}$	0			
2						1	0	1	$\underline{0}$	0	1	0	0
3	1	0	1	0	0	1	0	1	$\underline{0}$	0	1	0	$\longrightarrow$
	$\longrightarrow$	0	1	0	1	0	0	1	0	0			

....

Looking at the table above, we notice that if we define a substitution τ by $\tau(0) = 100$, $\tau(1) = 10$, then $\tau(v_n) = v_{n+1}$ for each $n \geq 0$.

Question. For which substitutions σ does there exist a substitution τ so that $\tau(v_n) = v_{n+1}$ for each $n \geq 0$, and what can we say about τ?

Theorem. Let σ be a Sturmian substitution with $\sigma(0)$ and $\sigma(1)$ Christoffel words. Then τ exists, and it is also a Christoffel substitution.

Example 4.

- $\sigma : \begin{cases} 0 \rightarrow 00101 \\ 1 \rightarrow 01 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 01011 \\ 1 \rightarrow 011 \end{cases}$

- $\sigma : \begin{cases} 0 \rightarrow 0010101 \\ 1 \rightarrow 01 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 0110111 \\ 1 \rightarrow 0111 \end{cases}$

- $\sigma : \begin{cases} 0 \rightarrow 0101011 \\ 1 \rightarrow 01011 \end{cases}$ gives $\tau : \begin{cases} 0 \rightarrow 000101 \\ 1 \rightarrow 001 \end{cases}$

From now on, assume σ is a Sturmian substitution that has incidence matrix with determinant 1, a fixed point starting with 0 and is NOT a Christoffel substitution. Put $M_\sigma = \begin{pmatrix} a & b \\ c+ag & d+bg \end{pmatrix}$ with $a+b > c+d$.

Denote by e the number of values left of the 0 position in $w_{\sigma(0)} = w_1$, by f the number of values left of the 0 position in $w_{\sigma(1)}$, by p the number of zeros in v_1 left of the underlined letter, and set $r = e + b(f - p - eg)$.

Example 5. Let $M_\sigma = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $\sigma(0) =$ 01001, $\sigma(1) =$ 010010101001. Then $g = 2$, $w_{\sigma(0)} =$ 2 0 3 1 4, $v_1 = 1\underline{0}101$ and $w_{\sigma(1)} =$ 9 2 7 0 5 10 3 8 1 6 11 4. Hence $e = 1$, $f = 3$, $p = 0$ and $r = 3$.

Definition. We denote by τ the substitution that has

$$M_\tau := \begin{pmatrix} c+d+dg & a+b+bg-(c+d+dg) \\ c+dg & a+bg-(c+dg) \end{pmatrix}$$

as incidence matrix, and which is such that

- if we cyclically shift $\tau(0)$ over r positions to the right, we get a Christoffel word,
- the $(r+1)$th letter of $\tau(0)$ is underlined,
- $\tau(1)$ equals the left $a+bg$ letters of $\tau(0)$.

Example 5 (Continued). We had $M_\sigma = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$, $\sigma(0) = 01001$, $\sigma(1) = 010010101001$ and $r = 3$.

We get $M_\tau = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ and $\tau(0) = 011\underline{0}10101$, $\tau(1) = 0110101$.

Theorem. Let σ be a Sturmian substitution that has an incidence matrix with determinant 1, that has a fixed point starting with 0, and that is not a Christoffel substitution. Let the Sturmian substitution τ and the central word v_n for $n \geq 1$ be defined as before. Then $\tau(v_n) = v_{n+1}$.

Remark. If the substitution τ has a fixed point starting with 0, we can apply the procedure again, and call the result ϕ. If $g = 0$ then $M_\phi = M_\sigma$, hence $\phi(0), \phi(1)$ are cyclic shifts of $\sigma(0), \sigma(1)$ respectively. In case σ is a Christoffel substitution, we get $\phi = \sigma$. This is not true in general.

Example 6. Let $M_\sigma = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $\sigma(0) = 0101101$, $\sigma(1) = 01101$. Then $e = 1$, $f = 1$, $p = 0$ and $r = 5$. We get $M_\tau = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ and $\tau(0) = 0100100$, $\tau(1) = 010$. Repeating this process, we get $e = 4$, $f = 1$, $p = 1$ and $r = 4$, which results in $\phi(0) = 1011010$, $\phi(1) = 10110$.

What if determinant is -1?

We can still form the central words v_n, except that for odd n, we need to reflect the central progressions w_n in the origin, before we construct v_n from w_n.

Since the substitution σ^2 has incidence matrix with determinant 1, it is clear that there exists a substitution τ_2 such that $v_{2n} = \tau_2(v_{2n-2})$. But as the following example shows, there does not need to exist a substitution τ such that $v_n = \tau(v_{n-1})$.

Example 7. Let $M_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma(0) = 001$, $\sigma(1) = 0$. Then we get the following table for v_n.

n													v_n				
1											1	1	$\underline{0}$				
2											1	1	$\underline{0}$	1	0	1	0
3	1	1	0	1	0	1	1	0	1	0	1	1	$\underline{0}$	1	0	1	0

$$\cdots\cdots$$

It is easy to check that there is no substitution τ such that $\tau(v_2) = v_3$.

However, this example suggests that if we define τ by $\tau(\underline{0}) = 11\underline{0}$, $\tau(1) = 10$, and $\tau(ab) = \tau(b)\tau(a)$ for every a, b in $\mathcal{A}$, then we have $v_n = \tau(v_{n-1})$.

An interesting question is if similar functions exist for all Sturmian substitutions that have an incidence matrix with determinant -1.