On the number of Pisot polynomials

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Definition 1 *Let $d \geq 1$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$: For $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ let*

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \ldots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1 a_1 + \ldots + r_d a_d$. We call $\tau_{\mathbf{r}}$ a shift radix system (SRS for short) if for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $k > 0$ with $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.

SRS form a common generalization of canonical number systems in residue class rings of polynomial rings as well as β-expansions of real numbers.

For $d \in \mathbb{N}$, $d \geq 1$ let

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic} \right\} \quad \text{a}$$

$$\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^d : \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\}.$$

$\mathcal{D}_d$ is strongly related to the set of contracting polynomials. In particular, let

$$\mathcal{E}_d(r) := \Big\{(r_1, \ldots, r_d) \in \mathbb{R}^d : X^d + r_d X^{d-1} + \cdots + r_1 \\ \text{has only roots } y \in \mathbb{C} \text{ with } |y| < r\Big\}.$$

Let $P(X) = X^d - b_1 X^{d-1} - \ldots - b_d \in \mathbb{Z}[X]$.

- If all but one root of P is located in the open unit disc then P is called a *Pisot polynomial*. Its dominant root is called *Pisot number*.

- If all but one root of P is located in the closed unit disc and at least one of them has modulus 1 then P is called a *Salem polynomial*. Its dominant root is called *Salem number*.

If P is a Pisot or Salem polynomial, we will denote its dominating root by β.

Let $\beta > 1$ and put $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Then each $\gamma \in [0, \infty)$ can be represented uniquely as a β-expansion by

$$\gamma = a_m \beta^m + a_{m-1} \beta^{m-1} + \cdots \tag{2}$$

with $a_i \in \mathcal{A}$ such that

$$0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n \tag{3}$$

holds for all $n \leq m$. Since the digits a_i are selected as large as possible, this representation is often called the *greedy expansion* of γ with respect to β.

K. Schmidt (1980) proved that in order to get ultimately periodic expansions for all $\gamma \in \mathbb{Q} \cap (0,1)$ it is necessary for β to be a Pisot or a Salem number.

Let Fin(β) be the set of positive real numbers having finite greedy expansion with respect to β. We say that $\beta > 1$ has property (F) if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).$$

It is shown by Frougny and Solomyak (1992) that (F) can hold only for Pisot numbers β. Akiyama, Brunotte, Pethő and Thuswaldner (2005) proved that property (F) is related to the SRS property.

Associated to Pisot and Salem numbers with periodic β-expansions and with property (F), respectively, we define for each $d \in \mathbb{N}$, $d \geq 1$ the sets

$$\begin{aligned} \mathcal{B}_d &:= \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : X^d - b_1 X^{d-1} - \ldots - b_d \\ &\qquad \text{is a Pisot or Salem polynomial}\} \quad \text{and} \\ \mathcal{B}_d^0 &:= \{(b_1, \ldots, b_d) \in \mathbb{Z}^d : X^d - b_1 X^{d-1} - \ldots - b_d \\ &\qquad \text{is a Pisot polynomial with property (F)}\}. \end{aligned}$$

We obviously have $\mathcal{B}_d^0 \subseteq \mathcal{B}_d$.

Let us consider the map $\psi : \mathcal{B}_d \to \mathbb{R}^{d-1}$. If $(b_1, \ldots, b_d) \in \mathcal{B}_d$ then let β be the dominant root of the polynomial

$$P(X) = X^d - b_1 X^{d-1} - \ldots - b_d.$$

Now let

$$\psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2),$$

where $r_2, \ldots, r_d$ are defined in a way that they satisfy the relation

$$X^d - b_1 X^{d-1} - \ldots - b_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \ldots + r_d).$$

As $(b_1, \ldots, b_d) \in \mathcal{B}_d$, the polynomial $X^{d-1} + r_2 X^{d-2} + \ldots + r_d$ has all its roots in the closed unit circle. Together with this implies that

$$\psi(\mathcal{B}_d) \subseteq \overline{\mathcal{D}_{d-1}}.$$

The above-mentioned relation between property (F) and SRS now reads as follows.

$$\psi(\mathcal{B}_d^0) \subseteq \mathcal{D}_{d-1}^0.$$

We show that $\psi(\mathcal{B}_d)$ and $\psi(\mathcal{B}_d^0)$ are excellent approximations of $\mathcal{D}_{d-1}$ and $\mathcal{D}_{d-1}^0$ respectively.

For $M \in \mathbb{N}_{>0}$ we set

$$\mathcal{B}_d(M) := \left\{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} \,:\, (M, b_2, \ldots, b_d) \in \mathcal{B}_d\right\} \tag{4}$$

and

$$\mathcal{B}_d^0(M) := \left\{(b_2, \ldots, b_d) \in \mathbb{Z}^{d-1} \,:\, (M, b_2, \ldots, b_d) \in \mathcal{B}_d^0\right\}. \tag{5}$$

With these notations we are able to state the following theorem.

Theorem 2 *We have*

$$\lim_{M \to \infty} \frac{|B_d(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}), \tag{6}$$

and

$$\lim_{M \to \infty} \frac{|B_d^0(M)|}{M^{d-1}} = \lambda_{d-1}(\mathcal{D}_{d-1}^0), \tag{7}$$

where λ_{d-1} denotes the $d-1$-dimensional Lebesgue measure.

Properties of two auxiliary mappings

For $M \in \mathbb{Z}$ let $\chi_M : \mathbb{R}^{d-1} \mapsto \mathbb{Z}^d$ such that if $\mathbf{r} = (r_2, \ldots, r_d)$ then $\chi_M(\mathbf{r}) = \mathbf{b} = (b_1, \ldots, b_d)$, where $b_1 = M$, $b_d = \lfloor r_d(M + r_2) + \frac{1}{2} \rfloor$ and

$$b_i = \lfloor r_i(M + r_2) - r_{i+1} + \frac{1}{2} \rfloor, i = 2, \ldots, d-1.$$

If $\mathbf{b} = (b_1, \ldots, d_d) \in \mathcal{B}_d$, then $\chi_{b_1}(\psi(\mathbf{b})) = \mathbf{b}$, i.e. χ_{b_1} is the inverse of ψ.

To prove the main theorem we need some properties of the sets

$$\mathcal{S}_d(M) = \chi_M(\overline{\mathcal{D}_{d-1}}) \quad \text{and} \quad \mathcal{S}_d^0(M) = \chi_M(\overline{\mathcal{D}_{d-1}^0})$$

and

$$\mathcal{S}_d = \cup_{M \in \mathbb{Z}} \mathcal{S}_d(M) \quad \text{and} \quad \mathcal{S}_d^0 = \cup_{M \in \mathbb{Z}} \mathcal{S}_d^0(M).$$

Our first Lemma shows that if $|M|$ is large enough then the polynomials associated to the elements of $\mathcal{S}_d$ behaves in some sense similar as Pisot or Salem polynomials.

Lemma 3 *Let $M \in \mathbb{Z}$, $(b_1, \ldots, b_d) \in \mathcal{S}_d(M)$ and $P(X) = X^d - b_1 X^{d-1} - \ldots - b_d$. There exist constants $c_1 = c_1(d), c_2 = c_2(d)$ such that if $|M|$ is large enough than $P(X)$ has a real root β for which the inequalities*

$$\begin{aligned} |\beta - b_1| &< c_1 & (8)\\ \left|\beta - b_1 - \frac{b_2}{b_1}\right| &< \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2}\right), & (9) \end{aligned}$$

hold.

There exists $(r_2,\ldots,r_d) \in \overline{\mathcal{D}_{d-1}}$ such that $\mathbf{b} = (b_1,\ldots,b_d) = \chi_M(r_2,\ldots,r_d)$.

It is easy to see that $|r_i| \leq 2^{d-1}$. Thus $b_i = Mr_i + O(1), i = 2,\ldots,d$.

Put $Q(X) = b_2X^{d-2} + \ldots + b_d$, i.e. let $P(X) = X^d - MX^{d-1} - Q(X)$. Then $P(M) = Q(M)$ and $P(M+t) = t(M+t)^{d-1} + Q(M+t)$. Assume that $M > 0$ and large enough and $Q(M) < 0$. As $|Q(M+t)| \leq d2^dM(M+t)^{d-2}$ we have $P(M+t) > 0$ provided $t \geq d2^d$. Thus $P(X)$ has a real root in the interval $(M, M+t)$ and (8) is proved with $c_1 = d2^d$.

The relation $P(\beta) = 0$ implies

$$\beta = b_1 + \frac{b_2}{\beta} + \frac{b_3}{\beta^2} + \ldots + \frac{b_d}{\beta^{d-1}}.$$

Thus

$$\beta - b_1 - \frac{b_2}{b_1} = \frac{(b_1 - \beta) b_2}{b_1 \beta} + \frac{b_3}{\beta^2} + \ldots + \frac{b_d}{\beta^{d-1}}.$$

using this expression, inequality (8) and the estimates $|b_i| = 2^d |M|, i = 2, \ldots, d$ we get

$$\begin{aligned} \left| \beta - b_1 - \frac{b_2}{b_1} \right| &\leq \frac{c_1 2^{d-1}}{|b_1| - c_1} + \frac{2^d |b_1|}{(|b_1| - c_1)^2} + \sum_{j=3}^{d-1} \frac{2^d |b_1|}{(|b_1| - c_1)^j} \\ &< \frac{c_2}{|b_1|} + O\left(\frac{1}{b_1^2} \right), \end{aligned}$$

which proves the second assertion of the Lemma.

Now we are in the position to extend the definition of ψ from the set $\mathcal{B}_d$ to S_d. If $(b_1, \ldots, b_d) \in \mathcal{S}_d$ and $|b_1|$ is large enough, then let β be the dominant root of the polynomial

$$P(X) = X^d - b_1 X^{d-1} - \ldots - b_d,$$

which exists by Lemma 3. Then let

$$\psi(b_1, \ldots, b_d) = (r_d, \ldots, r_2),$$

where the real numbers $r_2, \ldots, r_d$ are defined in a way that they satisfy the relation

$$X^d - b_1 X^{d-1} - \ldots - b_d = (X - \beta)(X^{d-1} + r_2 X^{d-2} + \ldots + r_d).$$

We also introduce an other mapping $\tilde{\psi} : \mathbb{Z}^d \mapsto \mathbb{Q}^{d-1}$ by

$$\tilde{\psi}(b_1,\ldots,b_d) = \left(\frac{b_d}{b_1 + \frac{b_2}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_2}{b_1}} + \frac{b_d}{b_1^2}, \ldots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2} \right).$$

The next lemma shows that if $(b_1,\ldots,b_d) \in \mathcal{S}_d$ then $\tilde{\psi}(b_1,\ldots,b_d)$ is a good approximation of $\psi(b_1,\ldots,b_d)$. We actually prove

Lemma 4 *Let* $(b_1,\ldots,b_d) \in \mathcal{S}_d$ *and assume that* $|b_1|$ *is large enough. Then*

$$\left|\tilde{\psi}(b_1,\ldots,b_d) - \psi(b_1,\ldots,b_d)\right|_\infty < \frac{c_3}{b_1^2} + O\left(\frac{1}{|b_1|^3}\right),$$

where c_3 *is depending only on* d.

In the next lemma we show that the set $\tilde{\psi}(\mathcal{S}_d)$ is lattice like. More precisely we prove

Lemma 5 *Let* $\mathbf{b} = (b_1, \ldots, b_d), \mathbf{b}' = (b'_1, \ldots, b'_d) \in \mathcal{S}_d$ *such that there exists a* $1 \leq j \leq d$ *such that* $b_i = b'_i, i \neq j$ *and* $b'_j = b_j + 1$. *Then*

$$|\tilde{\psi}(\mathbf{b})_k - \tilde{\psi}(\mathbf{b}')_k| = \begin{cases} 0, & \text{if } j > 2 \text{ and } k \neq d-j+1, d-j+2 \\ \frac{1}{|b_1|} + O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d-j+1 \text{ or } j = 2, k = d-1 \\ O(b_1^{-2}), & \text{if } j > 2 \text{ and } k = d-j+2 \text{ or } j = 2, k < d-1 \\ |b_{d-k+1}| \left(\frac{1}{b_1^2} + O(|b_1|^{-3} \right), & \text{if } j = 1. \end{cases}$$

A lemma on the roots of polynomials

Lemma 6 *Assume that all roots* $\alpha \in \mathbb{C}$ *of the polynomial* $P(x) = X^d + p_{d-1}X^{d-1} + \ldots + p_0 \in \mathbb{R}[X]$ *satisfy* $|\alpha| < \rho$. *Let* $\varepsilon > 0$ *and* $Q(x) = X^d + q_{d-1}X^{d-1} + \ldots + q_0 \in \mathbb{R}[X]$ *such that* $|p_i - q_i| < \varepsilon, i = 0, \ldots, d-1$. *Then for every root* α *of* $P(X)$ *there exists a root* β *of* $Q(X)$ *such that*

$$|\alpha - \beta| < \begin{cases} (d\varepsilon)^{1/d}, & \text{if } \rho \leq 1, \\ \left(\varepsilon \frac{\rho^d - 1}{\rho - 1}\right)^{1/d}, & \text{otherwise.} \end{cases}$$

Let $\alpha \in \mathbb{C}$ be a root of $P(X)$ and denote by $\beta_1, \ldots, \beta_d$ the roots of $Q(X)$. Then

$$Q(\alpha) - P(\alpha) = \sum_{i=0}^{d-1} \alpha^i (q_i - p_i) = \prod_{i=1}^{d} (\alpha - \beta_i).$$

We may assume without loss of generality $|\alpha - \beta_1| = \min_{1 \leq i \leq d} |\alpha - \beta_i|$. Then on one hand

$$\prod_{i=1}^{d} |\alpha - \beta_i| \geq |\alpha - \beta_1|^d$$

and on the other hand

$$\prod_{i=1}^{d} |\alpha - \beta_i| \leq \sum_{i=0}^{d-1} |\alpha|^i |q_i - p_i| \leq \varepsilon \sum_{i=0}^{d} \rho^i.$$

Comparing these inequalities we get the result.

The following Lemma is a immediate consequence of a theorem of Akiyama, Brunotte, Pethő and Thuswaldner (2005)

Lemma 7 *For every* $\varepsilon > 0$ *there exists* M_0 *such that if* $|M| > M_0$ *then*

$$\lambda_{d-1}\left(\mathcal{D}_{d-1} \setminus \mathcal{E}\left(1 - \sqrt[d]{\frac{d}{2|M|}}\right)\right) < \varepsilon$$

and

$$\lambda_{d-1}\left(\mathcal{D}^0_{d-1} \setminus \mathcal{E}\left(1 - \sqrt[d]{\frac{d}{2|M|}}\right)\right) < \varepsilon.$$

The next Lemma can be proved similarly as Lemma 4.7. of [**?**].

Lemma 8 *For every* $\varepsilon > 0$ *there exists* M_0 *such that if* $|M| > M_0$ *then*

$$\lambda_{d-1}\left(\mathcal{E}\left(1+\sqrt[d]{\frac{d}{2|M|}}\right)\setminus\mathcal{D}_{d-1}\right) < \varepsilon.$$

Proof of Theorem 2

Let $M > 0$ and put

$$W(\mathbf{x}, s) = \{\mathbf{y} \in \mathbb{R}^d \,:\, |\mathbf{x} - \mathbf{y}|_\infty \leq s/2\} \; (\mathbf{x} \in \mathbb{R}^d, s \in \mathbb{R})$$

and

$$\mathcal{W}_{d-1}(M) = \cup_{\mathbf{x} \in \mathcal{B}_d(M)} W(\psi(\mathbf{x}), M^{-1}).$$

Then we claim

$$\lambda_{d-1}(\mathcal{W}_{d-1}(M)) = \frac{|\mathcal{B}_d(M)|}{M^{d-1}} \left(1 + O\left(\frac{1}{M}\right)\right). \qquad (10)$$

Indeed, let $\mathbf{x}, \mathbf{y} \in \mathcal{B}_d(M)$ such that $\mathbf{x} - \mathbf{y} = \mathbf{e}_j$ for some $j \in \{2, \ldots, d\}$. Then by Lemmata 4 and 5

$$\begin{aligned} |\psi(\mathbf{x})_k - \psi(\mathbf{y})_k| &\leq |\psi(\mathbf{x})_k - \tilde{\psi}(\mathbf{x})_k + \tilde{\psi}(\mathbf{x})_k - \tilde{\psi}(\mathbf{y})_k + \tilde{\psi}(\mathbf{y})_k - \psi(\mathbf{y})_k| \\ &\leq \begin{cases} \frac{1}{M} + O\left(\frac{1}{M^2}\right), & \text{if } (j,k) = (2, d-1), \text{ or } j > 2, k = d - j + 1 \\ \left(\frac{1}{M^2}\right), & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$\lambda_{d-1}(W(\psi(\mathbf{x}), M^{-1}) \cap W(\psi(\mathbf{y}), M^{-1})) = O\left(\frac{1}{M^d}\right). \qquad (11)$$

As $\mathbf{x}$ has at most 2^d neighbors we get

$$\lambda_{d-1}\left(\bigcup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{B}_d(M)\\ \mathbf{x}\neq\mathbf{y}}} \Big(W(\psi(\mathbf{x}), M^{-1}) \cap W(\psi(\mathbf{y}), M^{-1})\Big)\right) = O\left(\frac{|\mathcal{B}_d(M)|}{M^d}\right)$$

and the claim is proved.

Now we are in the position to give lower estimate for $\lambda_{d-1}(\mathcal{D}_{d-1})$. Let $\mathbf{x} \in \mathcal{B}_d(M)$ such that $\psi(\mathbf{x}) \in \mathcal{E}\left(1 - \sqrt[d]{\frac{d}{2M}}\right) \subseteq \mathcal{D}_{d-1}$. Let $\mathbf{y} \in W(\psi(\mathbf{x}), M^{-1})$. Then $\rho(\psi(\mathbf{x})) < 1 - \sqrt[d]{\frac{d}{2M}}$ and as $|\psi(\mathbf{x}) - \mathbf{y}|_\infty \leq \frac{1}{2M}$ we get $\rho(\mathbf{y}) < 1$ by Lemma 6. Thus

$$\bigcup_{\substack{\mathbf{x} \in \mathcal{B}_d(M) \\ \rho(\psi(\mathbf{x})) < 1 - \sqrt[d]{\frac{d}{2M}}}} W(\psi(\mathbf{x}), M^{-1}) \subseteq \mathcal{D}_{d-1}. \tag{12}$$

Let $\varepsilon > 0$ and $M > M_0$, where M_0 is defined in Lemma 7. Then the number of $\mathbf{x} \in \mathcal{B}_d(M)$ such that $1 - \sqrt[d]{\frac{d}{2M}} \leq \rho(\psi(\mathbf{x})) \leq 1$ is at most $O(M^{d-1}\varepsilon)$ by Lemma 7 and by (11). Combining this with 11 and 12 we obtain the desired lower bound

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \geq \frac{|\mathcal{B}_d(M)|}{M^{d-1}} (1 - \varepsilon)\,. \tag{13}$$

To prove an upper bound we construct for every $\mathbf{r} = (r_d, \ldots, r_2) \in \mathcal{D}_{d-1}$ and M large enough a vector $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ such that $\psi(\mathbf{b})$ is lying near enough to $\mathbf{r}$.

Indeed put $\mathbf{b} = \chi_M(\mathbf{r})$ and consider

$$\tilde{\psi}(\mathbf{b}) = \left(\frac{b_d}{b_1 + \frac{b_2}{b_1}}, \frac{b_{d-1}}{b_1 + \frac{b_2}{b_1}} + \frac{b_d}{b_1^2}, \ldots, \frac{b_2}{b_1 + \frac{b_2}{b_1}} + \frac{b_3}{b_1^2} \right).$$

then we get

$$|\tilde{\psi}(\mathbf{b}) - \mathbf{r}|_\infty \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right).$$

Applying now Lemma 4 we obtain

$$|\psi(\mathbf{b}) - \mathbf{r}|_\infty \leq |\tilde{\psi}(\mathbf{b}) - \mathbf{r}|_\infty + |\psi(\mathbf{b}) - \tilde{\psi}(\mathbf{b})|_\infty \leq \frac{1}{2M} + O\left(\frac{1}{M^2}\right).$$

Thus by Lemma 6

$$\rho(\psi(\mathbf{b})) \le \rho(\mathbf{r}) + \sqrt[d]{\frac{d}{2M}} \le 1 + \sqrt[d]{\frac{d}{2M}}.$$

This means that if M is large enough then all but one roots of $P(X) = X^d - b_1 X^{d-1} - \ldots - b_d$ have absolute value at most $1 + \sqrt[d]{\frac{d}{2M}}$ and one root is close to M. We have further

$$\begin{aligned}\mathcal{D}_{d-1} &\subseteq \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d \\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+\sqrt[d]{\frac{d}{2M}})}} W(\psi(\mathbf{x}), M^{-1}) \\ &= \bigcup_{\mathbf{x}\in\mathcal{B}_d(M)} W(\psi(\mathbf{x}), M^{-1}) \cup \bigcup_{\substack{\mathbf{x}\in\mathbb{Z}^d \\ \psi(\mathbf{x})\in\mathcal{E}_{d-1}(1+\sqrt[d]{\frac{d}{2M}})\setminus\mathcal{E}(1)}} W(\psi(\mathbf{x}), M^{-1}).\end{aligned}$$

Let again $\varepsilon > 0$ and $M > M_0$, where M_0 is defined in Lemma 8.

Then Lemma 8 and (11) implies that the number of $\mathbf{x} \in \mathbb{Z}^d$ such that $\psi(\mathbf{x})$ is lying in $\mathcal{E}_{d-1}\left(1+\sqrt[d]{\frac{d}{2M}}\right)\setminus\mathcal{D}_{d-1}$ is at most $O(M^{d-1}\varepsilon)$, thus

$$\lambda_{d-1}(\mathcal{D}_{d-1}) \leq \frac{|\mathcal{B}_d(M)|}{M^{d-1}}\,(1+\varepsilon))\,.$$

Comparing this inequality with (13) we obtain the first statement of Theorem 2.