FRACTAL CRYSTALLOGRAPHIC TILINGS

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Introduction

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- Question: when is T homeomorphic to a closed disk?
- Results: criteria involving the configuration of the neighbors of T in the tiling.

Crystallographic tiling

- If T is a compact set with $T = \overline{T^o}$, Γ a family of isometries of $\mathbb{R}^2$ such that $\mathbb{R}^2 = \bigcup_{\gamma\in\Gamma} \gamma(T)$ and the $\gamma(T)$ do not overlap, we say that T tiles $\mathbb{R}^2$ by Γ.

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- $\Gamma \le \mathrm{Isom}(\mathbb{R}^2)$ is a crystallographic group if $\Gamma \simeq \mathbb{Z}^2 \ltimes \{id, r_2, \ldots, r_d\}$ with $r_2, \ldots, r_d$ isometries of finite order greater than 2.

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A crystallographic reptile with respect to $(\Gamma, \mathcal{D}, g)$ is a set $T \subset \mathbb{R}^2$ such that T tiles $\mathbb{R}^2$ by Γ and

$$g(T) = \bigcup_{\delta \in \mathcal{D}} \delta(T).$$

An example of crystile

We consider

- the group $p3 = \{\ a^i b^j r^k,\ \ i,j \in \mathbb{Z}\ ,\ k \in \{0,1,2\}\ \}$ where

$$\begin{array}{rcl} a(x,y) & = & (x+1,y) \\ b(x,y) & = & \left(x+1/2, y+\sqrt{3}/2\right), \\ r & = & \mathrm{rot}[0, 2\pi/3] \end{array}$$

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- the digit set $\{id, ar^2, br^2\}$,

- the map $g(x,y) = \sqrt{3}(y,-x)$.

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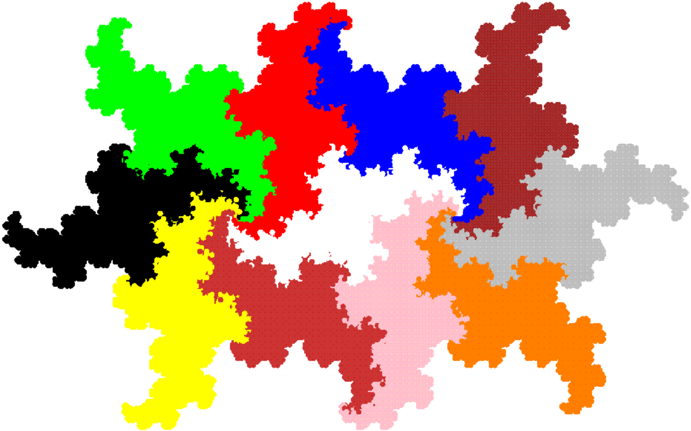

Figure: Terdragon T defined by $g(T) = T \cup ar^2(T) \cup br^2(T)$.

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Therefore, each $x\in T$ has an adress

$$x = (\delta_1\ \delta_2\ \dots)\,.$$

Known results

- [Gelbrich - 1994] Two crystiles $(T;\Gamma,\mathcal{D},g)$ and $(T';\Gamma',\mathcal{D}',g')$ are isomorphic if there is an affine bijection $\phi : T \to T'$ preserving the pieces of all levels. There are at most finitely many isomorphy classes of disk-like plane crystiles with k digits $(k \geq 2)$.

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- [Luo, Rao, Tan - 2002] T connected self-similar tile with $T^o \neq \emptyset$ is disk-like whenever its interior is connected.

- [Bandt, Wang - 2001] Criterion of disk-likeness for lattice tiles in terms of the number of neighbors of the central tile.

Neighbors

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- The boundary of T is:

$$\partial T = \bigcup_{\gamma \in \mathcal{S}} T \cap \gamma(T).$$

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- the vertices are the $\gamma \in \mathcal{S}$,
- there is an edge $\gamma \xrightarrow{\delta_1|\delta_1'} \gamma_1 \in G(\mathcal{S})$ iff

$$\gamma\ g^{-1}\delta_1' = g^{-1}\delta_1\ \gamma_1$$

with $\gamma, \gamma_1 \in \mathcal{S}$ and $\delta_1, \delta_1' \in \mathcal{D}$.

Boundary characterization

Theorem

Let $\delta_1, \delta_2, \ldots$ a sequence of digits and $\gamma \in \mathcal{S}$. Then the following assertions are equivalent.

- $x = (\delta_1 \ \delta_2 \ \ldots) \ \in T \cap \gamma(T)$.
- *There is an infinite walk in $G(\mathcal{S})$ of the shape:*

$$\gamma \xrightarrow{\delta_1|\delta_1'} \gamma_1 \xrightarrow{\delta_2|\delta_2'} \gamma_2 \xrightarrow{\delta_3|\delta_3'} \ldots \qquad (1)$$

for some $\gamma_i \in \mathcal{S}$ and $\delta_i' \in \mathcal{D}$.

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for some $\gamma_i \in \mathcal{S}$ and $\delta_i' \in \mathcal{D}$.

Remark. The set of neighbors $\mathcal{S}$ and the boundary graph $G(\mathcal{S})$ can be obtained algorithmically for given data $(\Gamma, \mathcal{D}, g)$.

Neighbor and Adjacent neighbor graphs

The neighbor graph of a crystallographic tiling is the graph G_N with

- vertices $\gamma \in \Gamma$
- edges $\gamma - \gamma'$ if $\gamma(T) \cap \gamma'(T) \neq \emptyset$, *i.e.*, $\gamma' \in \gamma\mathcal{S}$.

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Adjacent neighbors: γ, γ' with $\gamma(T) \cap \gamma'(T)$ contains a point of $(\gamma(T) \cup \gamma'(T))^o$. $\mathcal{A}$ denotes the set of adjacent neighbors of id. It can be obtained with the help of $G(\mathcal{S})$.

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The adjacent neighbor graph of a crystallographic tiling is the graph G_A with

- vertices $\gamma \in \Gamma$
- edges $\gamma - \gamma'$ if $\gamma(T)$ and $\gamma'(T)$ are adjacent, *i.e.*, $\gamma' \in \gamma\mathcal{A}$.

G_A and G_N for the $p3$ example

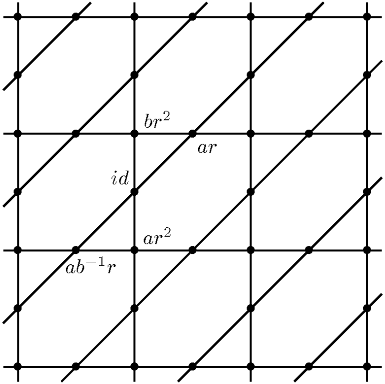

Figure: Adjacent neighbor graph for the Terdragon.

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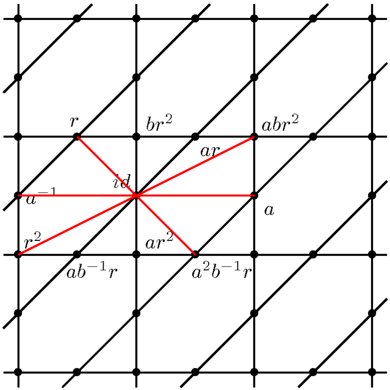

Figure: G_A and the neighbors of the identity.

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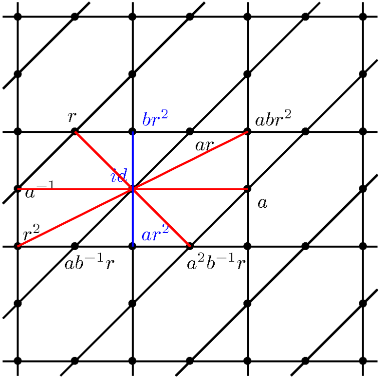

Figure: G_A and the neighbors of the identity. In blue: the digits.

General criterion of disk-likeness

Theorem (with Luo J. and J.-M. Thuswaldner)

Let T be a planar crystallographic reptile with respect to the group Γ. Then T is disk-like iff the following three conditions hold:

- (i) *the adjacent graph G_A is a connected planar graph,*
- (ii) *the digit set $\mathcal{D}$ induces a connected subgraph in G_A,*
- (iii) *G_N can be derived from G_A by joining each pair of vertices in the faces of G_A.*

Criteria on the shape of the neighbor set

- [Grünbaum, Shephard - 1987] There are finitely many possible sets $(\mathcal{S}, \mathcal{A})$ such that a disk-like crystallographic tile admits $(\mathcal{S}, \mathcal{A})$ as sets of neighbors and adjacent neighbors.

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- Reciprocal statement for crystallographic reptiles ?

Lattice case

If $\mathcal{F}$ is a subset of $\mathcal{S}$, the digit set $\mathcal{D}$ is said to be $\mathcal{F}$-connected if for every pair (δ, δ') of digits there is a sequence

$$\delta \xrightarrow[\delta^{-1}\delta_1 \in \mathcal{F}]{} \delta_1 \xrightarrow[\delta_1^{-1}\delta_2 \in \mathcal{F}]{} \delta_2 \rightarrow \cdots \rightarrow \delta_{n-1} \xrightarrow[\delta_{n-1}^{-1}\delta' \in \mathcal{F}]{} \delta'$$

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with $\delta_i \in \mathcal{D}$.

Theorem (Bandt, Wang - 2001)

Let T be a self-affine lattice plane tile with digit set $\mathcal{D}$.

(1) *Suppose that the neighbor set $\mathcal{S}$ of T has not more than six elements. Then T is disk-like iff $\mathcal{D}$ is $\mathcal{S}$-connected.*

(2) *Suppose that the neighbor set $\mathcal{S}$ of T has exactly the eight elements $\{a^{\pm 1}, b^{\pm 1}, (ab)^{\pm 1}, (ab^{-1})^{\pm 1}\}$, where a and b denote two independent translations. Then T is disk-like iff $\mathcal{D}$ is $\{a^{\pm 1}, b^{\pm 1}\}$-connected.*

$p2$ case

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Theorem (with Luo J.)

Let T be a crystile that tiles the plane by a $p2$-group and $\mathcal{D}$ the corresponding digit set.

(1) *Suppose that the neighbor set $\mathcal{S}$ of T has six elements. Then T is disk-like iff $\mathcal{D}$ is $\mathcal{S}$-connected.*

(2) *Suppose that the neighbor set $\mathcal{S}$ of T has exactly the seven elements $\{b, b^{-1}, c, bc, a^{-1}c, a^{-1}bc, a^{-1}b^{-1}c\}$, where a, b are translations and c is a π-rotation. Then T is disk-like iff $\mathcal{D}$ is $\{b, b^{-1}, c, bc, a^{-1}c\}$-connected.*

(3) *Similar results as (2) hold if $\mathcal{S}$ has 8 elements or 12 elements.*

Example for $p2$ case

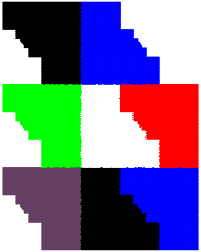

Figure: $g(x, y) = (y, 3x + 1)$, $\mathcal{D} = \{id, b, c\}$.

Open questions

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- Other topological properties (fundamental group).