Bernoulli convolutions associated with some algebraic numbers

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Outline

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Introduction

Fix $\lambda > 1$. Consider the random series

$$F_\lambda = \sum_{n=0}^{\infty} \epsilon_n \lambda^{-n},$$

where $\{\epsilon_n = \epsilon_n(\omega)\}$ is a sequence of i.i.d random variables taking the values 0 and 1 with prob. $(1/2, 1/2)$.

Let μ_λ be the distribution of F_λ, i.e.,

$$\mu_\lambda(E) = \textbf{Prob}(F_\lambda \in E), \qquad \forall \textbf{ Borel } E \subset \mathbb{R}$$

The measure μ_λ is called the **Bernoulli convolution** associated with λ. It is supported on the interval $[0, \lambda/(\lambda-1)]$.

The following are some basic properties:

- μ_λ is the infinite convolution of $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\lambda^{-n}}$.
- Let $\widehat{\mu_\lambda}(\xi) = \int \exp(i2\pi\xi x)\, d\mu_\lambda(x)$ be the Fourier transform of μ_λ. Then

$$|\widehat{\mu_\lambda}(\xi)| = \prod_{n=0}^{\infty} \left|\cos(\pi\lambda^{-n}\xi)\right|.$$

- Self-similar relation:

$$\mu_\lambda(E) = \frac{1}{2}\mu_\lambda\left(\phi_1^{-1}(E)\right) + \frac{1}{2}\mu_\lambda\left(\phi_2^{-1}(E)\right),$$

where $\phi_1(x) = \lambda^{-1}x$ and $\phi_2(x) = \lambda^{-1}x + 1$.

- Density function $f(x) = \frac{d\mu_\lambda(x)}{dx}$ (if it exists) satisfies the refinement equation

$$f(x) = \frac{\lambda}{2} f(\lambda x) + \frac{\lambda}{2} f(\lambda x - \lambda).$$

- (Alexander & Yorke, 1984): μ_λ is the projection of the SRB measure of the Fat baker transform $T_\lambda : [0,1]^2 \to [0,1]^2$, where

$$T_\lambda(x,y) = \begin{cases} (\lambda^{-1}x, 2y), & \text{if } 0 \le y \le 1/2 \\ (\lambda^{-1}x + 1 - \lambda^{-1}, 2y - 1), & \text{if } 1/2 < y \le 1 \end{cases}$$

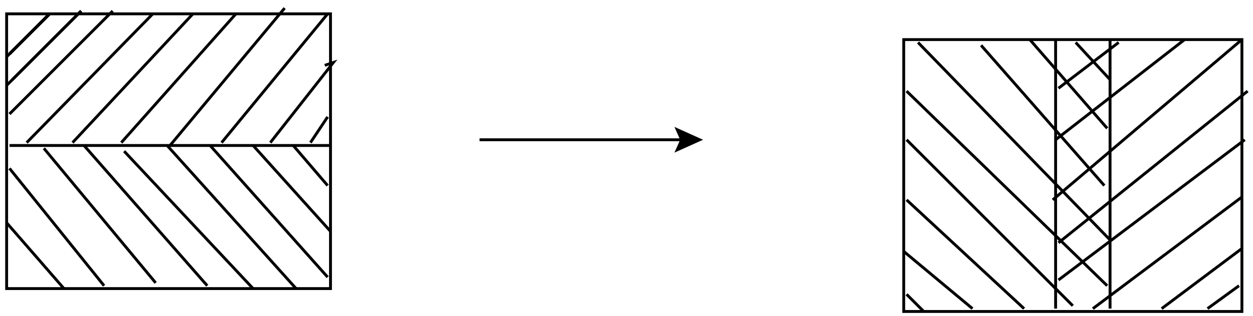

Figure 1: The Fat Baker transformation T_λ

Classical questions

- **For which $\lambda \in (1,2)$, μ_λ is absolutely continuous?**
- **If abs. cont., how smooth is the density $\frac{d\mu_\lambda}{dx}$?**
- **If singular, how to describe the local structure and singularity?**
- **Does μ_λ have some kind of Gibbs property? Does the multifractal formalism holds for μ_λ?**

Remark

- If $\lambda > 2$, μ_λ is a Cantor measure and thus is singular.
- If $\lambda = 2$, μ_λ is just the uniform distribution on $[0,2]$.
- For all $\lambda > 1$, μ_λ is either absolutely continuous or singular (Jessen & Wintner, 1935).

Partial answers:

- (Erdös, 1939): For the golden ratio $\lambda = \frac{\sqrt{5}+1}{2}$, μ_λ is singular

  In fact Erdös showed that for the golden ratio, $\widehat{\mu_\lambda}(\xi) \not\to 0$ as $\xi \to \infty$ using the key algebraic property: $\text{dist}(\lambda^n, \mathbb{Z}) \to 0$ exponentially. The same property holds when λ is a **Pisot number** (i.e., an algebraic integer whose conjugates are all inside the unit disc).

  Remark: Using Erdös' method one can not find new parameter λ for which μ_λ is singular. Since Salem (1963) proved that the property $\widehat{\mu_\lambda}(\xi) \not\to 0$ as $\xi \to \infty$ implies that λ is a Pisot number.

- For a family of explicit algebraic integers λ called **Garcia numbers** (namely, a real algebraic integer $\lambda > 1$ such that all its conjugates are larger than 1 in modulus, and their product together with λ equals ± 2), *e.g.*, $\lambda = \sqrt[n]{2}$ or the largest root of $x^n - x - 2$, μ_λ is absolutely continuous. (Garcia, 1965).

  $\exists C > 0$ s.t for $I = i_1 \dots i_n, J = j_1 \dots j_n \in \{0,1\}^n$ with $I \neq J$,

$$\left| \sum_{k=1}^{n} (i_k - j_k) \lambda^{-k} \right| > C \cdot 2^{-n}.$$

- (Erdös, 1940): There exists a very small number $\delta \approx 2^{2^{-10}} - 1$ such that μ_λ is absolutely continuous for a.e $\lambda \in (1, 1+\delta)$.
- (Solomyak, 1995, Ann Math.): **For a.e. $\lambda \in (1,2)$, μ_λ is absolutely continuous with density $\frac{d\mu_\lambda}{dx} \in L^2$.**

Open Problems:

- Are the Pisot numbers the only ones for which μ_λ are singular?
- Can we construct explicit numbers other than Garcia numbers for which μ_λ are absolutely continuous?

Characterizing singularity (Pisot numbers)

- Golden ratio case:
  (entropy, Hausdorff dimension, local dimensions, multifractal structure of μ_λ) has been considered by many authors, e.g., Alexander-Yorke (1984), Ledrappier-Porzio(1992), Lau-Ngai(1998), Sidorov-Vershik(1998). F. & Olivier(2003).
- Pisot numbers:
  (Lalley(1998): $\dim_H \mu_\lambda$ =Lyapunov exponent of random matrice).

- Dynamical structures corresponding to Pisot numbers

Theorem (F., 2003, 2005)
The support of μ_λ can be coded by a subshift of finite type, and

$$\mu_\lambda([i_1 \dots i_n]) \approx \|M_{i_1} \dots M_{i_n}\|$$

where $\{M_i\}$ is a finite family of non-negative matrices.

The above result follows from the finiteness property of Pisot numbers:

$$\# \left\{ \sum_{k=1}^{n} \epsilon_k \lambda^k : \; n \in \mathbb{N}, \epsilon_k = 0, \pm 1 \right\} \cap [a, b] < \infty$$

for any a, b.

For some special case, e.g., when λ is the largest root of $x^k - x^{k-1} - \ldots - x - 1$, the above product of matrices is degenerated into product of scalars; and locally μ_λ can be viewed as **a self-similar measure with countably many non-overlapping generators**. As an application, some explicit dimension formulae are obtained for μ_λ.

Non-smoothness.

Theorem (Kahane, 1971)

$\frac{d\mu_\lambda}{dx} \notin C^1$ for Salem numbers λ (since there are no $\alpha > 0$ such that $\widehat{\mu_\lambda}(\xi) = O(|\xi|^{-\alpha})$ at infinity)

Problem : Is there non-Pisot number for which $\frac{d\mu_\lambda}{dx} \notin L^2$?

Theorem (F. & Wang, 2004) Let λ_n be the largest root of $x^n - x^{n-1} - \ldots - x^3 - 1$. Then for any $n \geq 17$, λ_n is non-Pisot and $\frac{d\mu_{\lambda_n}}{dx} \notin L^2$.

Our result hints that perhaps μ_{λ_n} is singular.

Theorem (F. & Wang, 2004) Let λ_n be the largest root of $x^n - x^{n-1} - \ldots - x + 1$, $n \geq 4$, (λ_n are Salem numbers). Then for any $\epsilon > 0$, $\frac{d\mu_{\lambda_n}}{dx} \notin L^{3+\epsilon}$ when n is large enough.

Conjecture: there is a set Λ dense in $(1, 2)$ such that $\frac{d\mu_{\lambda}}{dx} \notin L^2$ for $\lambda \in \Lambda$?

Smoothness

Problem : For which λ, the density $\frac{d\mu_\lambda}{dx}$ is a piecewise polynomial?

Answer: If and only if $\lambda = \sqrt[n]{2}$.
(Dai, F. & Wang, 2006)

Problem: For which λ, $\widehat{\mu_\lambda}(\xi)$ has a decay at ∞? i.e., there exists $\alpha > 0$ such that $\widehat{\mu_\lambda}(\xi) = O(|\xi|^{-\alpha})$.

Remark: If $\widehat{\mu_\lambda}(\xi)$ has a decay at ∞, then $\mu_{\lambda^{1/n}}$ has a C^k density if n is large enough.

Theorem (Dai, F. & Wang, to appear in JFA): If λ is a Garcia number, then $\widehat{\mu_\lambda}(\xi)$ has a decay at ∞.

Problem :

Is μ_λ absolutely continuous for $\lambda = \frac{3}{2}$?

It is still open. But it is true for the distribution of the random series

$$\sum_{n=0}^{\infty} \epsilon_n \lambda^{-n},$$

where $\epsilon_n = 0, 1, 2$ with probability $1/3$, and $\lambda = \frac{3}{2}$

(Dai, F. & Wang)

Moreover for any rational number $\lambda \in (1,2)$, and $k \in \mathbb{N}$, we can find a digit set D of integers and a probability vector $\mathbf{p} = (p_1, \ldots, p_{|D|})$ such that the distribution of the random series

$$\sum_{n=0}^{\infty} \epsilon_n \lambda^{-n}$$

has a C^k density function, where ϵ_n is taken from D with the distribution $\mathbf{p}$.

However, the above result is not true if λ is a non-integral Pisot number, e.g., $\frac{\sqrt{5}+1}{2}$

Gibbs properties

Is μ_λ equivalent to some invariant measure of a dynamical system?

(Sidorov & Vershik, 1999): For $\lambda = \frac{\sqrt{5}+1}{2}$, μ_λ is equivalent to an ergodic measure ν of the map $T_\lambda : [0,1] \to [0,1]$ defined by

$$x \to \lambda x \pmod 1$$

Question by S&V: Is the corresponding measure ν a Gibbs measure?

Answer: It is a kind of weak-Gibbs measure. (Olivier & Thomas)

Theorem (F., to appear in ETDS)

For any $\lambda > 1$, μ_λ has a kind of Gibbs property as follows:

For $q > 1$, there exists a measure $\nu = \nu_q$ such that for any x

$$\nu(B_r(x)) \preceq r^{-\tau(q)}(\mu_\lambda(B_r(x)))^q.$$

As a result, μ_λ always partially satisfies the multifractal formalism.

In particular, if λ is a Salem number, we have

$$\nu(B_r(x)) \preceq C_r r^{-\tau(q)}(\mu_\lambda(B_r(x)))^q$$

for all $q > 0$, where $\log c_r / \log r \to 0$ as $r \to 0$.

Applications of Abs-Continuity property

(1) Dimension estimates of some affine graphs:

- Let $W(x)$ denote the Weirestrass function

$$W(x) = \sum_{n=0}^{\infty} \lambda^{-n} \cos(2^n x)$$

It is an open problem to determine if or not the Hausdorff dimension of the graph of W is equal to its box dimension (the latter equals $2 - \log \lambda / \log 2$)

- Consider the same question for the graph of the Rademacher series

$$F(x) = \sum_{n=0}^{\infty} \lambda^{-n} R(2^n x)$$

where R is a function of period 1, taking value 1 on $[0, 1/2)$ and 0 on $[1/2, 1)$.

Pryzycki & Urbanski (1989) showed that if μ_λ is absolutely continuous then the graph of F has the same Hausdorff dimension and box counting dimension. Moreover they show that if λ is a Pisot number, then the Hausdorff dimension is strictly less than its box counting dimension (the latter equals $2 - \log \lambda / \log 2$). The explicit value is obtained for some special Pisot numbers (F. 2005).

Applications of Abs-Continuity property

(2) Absolute continuity of the SRB measure of the Fat Baker transform.

Alexander & Yorke (1984) showed that if μ_λ is abs cont., then so is the corresponding SRB measure of the Fat Baker transform T_λ.