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A Digital Description of the Fundamental Group of Fractals I

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Project: Metric and Topological Aspects of Number Theoretical Problems

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Analytic Combinatorics and Probabilistic Number Theory

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Some references

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Digital representation of the Sierpiński gasket $\triangle$

by sequences with digits $\{0, 1, 2\}$

dyadic points: belong to 2 subtriangles in $\triangle_n$, the smallest such n is the order of the dyadic point

dyadic points $P \neq (0), (1), (2)$ have 2 representations as sequences in $\{0, 1, 2\}^{\mathbb{N}}$
e.g. $P = (0, 1, 2, 2, \ldots) = (0, 2, 1, 1, \ldots) =: (0, 1|2)$

dyadic points correspond to sequences which are eventually constant

D_n: dyadic points of order $\leq n$

generic points have a unique representation

Symbolic representation of loops in $\triangle$

$\omega : \ [0,1] \to \triangle, \quad \omega(0) = \omega(1) = (0)$

fixed approximation level n:

$\{\omega^{-1}(P) | P \in D_n\}$ is a finite family of disjoint closed set $\subseteq [0,1] \to$ separated family of sets

$\omega \mapsto \underline{\sigma_n(\omega)}$: contains the (finite!) sequence of dyadic points of order $\leq n$ that ω "passes"

$\sigma_n(\omega)$ is a finite word over the alphabet D_n

Frame for $(\sigma_n(\omega))_{n\in\mathbb{N}}$

$\underline{S_n:}$ the set of all "admissible" words ω_n over the alphabet D_n, i.e.

1. ω_n starts and ends with (0)
2. consecutive letters in ω_n are neighboring dyadic points in $\triangle_n$

$\underline{(S_n, \cdot):}$ semigroup where $\cdot$ is concatenation of words and one intermediate (0) is cancelled

$\underline{\gamma_n : \ S_n \to S_{n-1}:}$ γ_n deletes all points of order n and cancels out repetitions of points

γ_n is a semigroup homomorphism

$$\downarrow$$

$\varprojlim S_n$ inverse limit of semigroups S_n

Proposition. Let $\omega : [0,1] \to \triangle$ be a loop in $\triangle$. Then $(\sigma_n(\omega))_{n\in\mathbb{N}} \in \varprojlim S_n$.

Reduction process reflecting homotopy

reduced words in S_n: do not contain subwords of the form $\underline{PQP}$, or $\underline{PQR}$, where P, Q, R belong to the same subtriangle of $\triangle_n$

$\underline{G_n}$: the set of all reduced words over the alphabet D_n

$\underline{\text{Red}_n : \ S_n \to G_n}$: reduces subwords

$$\begin{cases} PQP \to P, & \text{and} \\ PQR \to PR & (P, Q, R \text{ in the same subtriangle}) \end{cases}$$

until word is reduced

- Red_n well defined

- $\text{Red}_n(\omega_n)$ canonical representative of the homotopy class of the elementary path corresp. to ω_n in $\triangle_n$

multiplication $*$ in G_n:

$$\omega_n * \bar{\omega}_n := \text{Red}_n(\omega_n \cdot \bar{\omega}_n)$$

Proposition. $(G_n, *)$ is isomorphic to the fundamental group of $\triangle_n$.

$$\delta_n : \begin{cases} G_n & \rightarrow & G_{n-1} \\ \omega_n & \mapsto & \text{Red}_{n-1}(\gamma_n(\omega_n)) \end{cases}$$ is a group homomorphism

$$\downarrow$$

$\varprojlim G_n$ inverse limit of groups

Proposition. The Čech homotopy group of $\triangle$ is isomorphic to $\varprojlim G_n$.

the following diagram commutes:

$$\begin{array}{ccc} S_n & \xrightarrow{\gamma_n} & S_{n-1} \\ \downarrow \text{Red}_n & & \text{Red}_{n-1} \downarrow \\ G_n & \xrightarrow{\delta_n} & G_{n-1} \end{array}$$

$$\begin{array}{ccc} S(\triangle) & \xrightarrow{\sigma} & \varprojlim S_n \\ \downarrow [\,.\,] & & \text{Red} \downarrow \\ \pi(\triangle) & \xrightarrow{\varphi} & \varprojlim G_n \end{array}$$

$(S(\triangle), \cdot)$: groupoid of loops in $\triangle$ with concatenation $\cdot$

$[\omega]$ homotopy class of ω

$\sigma(\omega) := (\sigma_n(\omega))_{n\in\mathbb{N}}$

$\text{Red}((\omega_n)_{n\in\mathbb{N}}) := (\text{Red}_n(\omega_n))_{n\in\mathbb{N}}$

$\varphi([\omega]) := (\text{Red}_\text{n}(\sigma_n(\omega)))_{n\in\mathbb{N}}$

- φ is injective (Eda/Kawamura 1998), i.e. $\pi(\triangle)$ is a subgroup of $\varprojlim G_n$

- $\underline{\varphi \text{ is not surjective:}}$

Example 1.
$$\begin{aligned} \omega_1 &= (0) \\ \omega_2 &= C_0 C_1 C_0^{-1} \\ \omega_3 &= C_0 C_1 C_0^{-1} C_2 \\ \omega_4 &= C_0 C_1 C_0^{-1} C_2 C_0 C_3 C_0^{-1} \\ &\dots \end{aligned}$$

$(\omega_n)_{n\in\mathbb{N}} \in \varprojlim G_n$, but $(\omega_n)_{n\in\mathbb{N}} \notin \text{range}(\varphi)$:

a loop ω in $\triangle$ with $\varphi([\omega]) = (\omega_n)_{n\in\mathbb{N}}$ has to pass the cycle C_0 infinitely often

- $\underline{\text{range}(\varphi)} = \text{range}(\varphi \circ [\,.\,]) = \underline{\text{range}(\text{Red} \circ \sigma)}$

- $\underline{\sigma \text{ is not surjective:}}$

Example 2.

$\omega_1 = (0)(0|1)(0)$
$\omega_2 = (0)(0,0|1)(0|1)(1,0|1)(0|1)(0,0|1)(0)$
$\omega_3 = (0)(0,0,0|1)(0,0|1)\dots(1,1,0|1)\dots(0)$
$\dots$

- graph associated to $(\omega_n)_{n\in\mathbb{N}} \in \varprojlim S_n$:

  - every branch corresponds to a dyadic point
  - there is total order on the branches
  - this order is dense
  - every Dedekind cut in the set of branches converges to a point in the Sierpiński gasket

Range and kernel of σ

Theorem. $(\omega_n)_{n\in\mathbb{N}} \in \varprojlim S_n$ is in the range of σ if and only if every irrational Dedekind cut in the set of branches of the graph associated to $(\omega_n)_{n\in\mathbb{N}}$ converges to a generic point in $\triangle$.

Theorem. For ω and $\bar{\omega}$ in $S(\triangle)$ we have $\sigma(\omega) = \sigma(\bar{\omega})$ if and only if ω and $\bar{\omega}$ have a common re-parametrization, i.e. there exist $\alpha, \beta : [0,1] \to [0,1]$ monotonously increasing and surjective such that $\omega \circ \alpha = \bar{\omega} \circ \beta$.

Main Theorem. An element $(\omega_n)_{n\geq 0}$ of $\varprojlim G_n$ is in $\varphi(\pi(\triangle))$ if and only if for all $k \geq 0$ the sequence

$$(\gamma_{nk}(\omega_n))_{n\geq k}$$

stabilizes, where $\gamma_{nk} = \gamma_{k+1} \circ \ldots \circ \gamma_n$.