On some distributions related to digital expansions

Ligia-Loretta Cristea
Institut für Finanzmathematik, Johannes Kepler Universität Linz
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joint work with **Helmut Prodinger**

Overview

- The binomial distribution and its moments
- The Gray code distribution and its moments

The binomial distribution and its moments

The binomial measure. Definitions and notations

Definition

(Okada, Sekiguchi, Shiota, 1995)
Let $0 < r < 1$ and $I = I_{0,0} = [0, 1]$,

$$I_{n,j} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \text{ for } j = 0, 1, \ldots, 2^n - 2, \qquad I_{n,2^n-1} = \left[\frac{2^n - 1}{2^n}, 1\right],$$

for $n = 1, 2, 3, \ldots$. The binomial measure μ_r is a probability measure on I uniquely determined by the conditions

$$\mu_r(I_{n+1,2j}) = r\mu_r(I_{n,j}), \qquad \mu_r(I_{n+1,2j+1}) = (1 - r)\mu_r(I_{n,j}),$$

for $n = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, 2^n - 1$.

$I_{n,j}$ = *elementary intervals of level* n, $n \in \{1, 2, 3, \ldots\}$, $j \in \{0, \ldots, 2^n - 1\}$

The binomial measure. Definitions and notations

Notations

- $\mathcal{W}$ = the set of all infinite words over the alphabet $D = \{0, 1\}$
- $\mathcal{W}_m$ = the set of all words of length m ($m \geq 1$) over the alphabet D.
- For every word $\omega \in \mathcal{W}$, $\omega = \omega_1\omega_2\ldots\omega_n\ldots$ we define its *value*

$$val(\omega) = \sum_{i \geq 1} \omega_i \cdot 2^{-i}.$$

  Thus we assign to every infinite word $\omega = \omega_1\omega_2\ldots$ the binary fraction $0.\omega_1\omega_2\ldots$.

- Analogously we define the value of any word of $\mathcal{W}_m$

Remark. In the case of choosing in a random way (with respect to μ_r) a word $\omega \in \mathcal{W}$, we have

$$\mathbb{P}(\omega_k = 0) = \mathbb{P}(\omega_k = 1) = \frac{1}{2}, \text{ for } k = 1, 2, \ldots,$$

These probabilities depend neither on the parameter r nor on k.

The moments of the binomial distribution

We study the moments of the function *val* with respect to the distribution defined by μ_r.

- M_n the moment of order n

$$M_n = \sum_{\omega \in \mathcal{W}} \mu_r(\omega) \cdot (val(\omega))^n.$$

Let

$$M_n{}^m = \sum_{\omega \in \mathcal{W}_m} \mu_r(\omega) \cdot (val(\omega))^n.$$

We have $M_n = \lim_{m \to \infty} M_n{}^m$.

- It is easy to verify that

$$val(d\omega) = d \cdot 2^{-1} + 2^{-1} \cdot val(\omega).$$

The moments of the binomial distribution

Notation: $\mathcal{W}_m^k$ = the set of words of $\mathcal{W}_m$ containing exactly k times the character 0.

We have

$$M_n^m = \sum_{k=0}^{m} r^k (1-r)^{m-k} \sum_{\omega \in \mathcal{W}_m{}^k} (val(\omega))^n.$$

By analysing the first character of the words occurring in the last sum we get

$$M_n^m = r \cdot \frac{1}{2^n} \sum_{k=0}^{m-1} r^k (1-r)^{m-1-k} \sum_{\omega \in \mathcal{W}_{m-1}^k} (val(\omega))^n$$

$$+ (1-r) \cdot \frac{1}{2^n} \sum_{k=0}^{m-1} r^k (1-r)^{m-1-k} \sum_{\omega \in \mathcal{W}_{m-1}^k} (1 + val(\omega))^n$$

$$= \frac{1}{2^n} M_n^{m-1} + (1-r) \cdot \frac{1}{2^n} \sum_{j=0}^{n-1} \binom{n}{j} M_j^{m-1}.$$

Theorem

The moments of the binomial distribution μ_r *satisfy the relations:*

$$M_0 = 1,$$

$$M_n = \frac{r}{2^n} M_n + \frac{1-r}{2^n} \sum_{j=0}^{n} \binom{n}{j} M_j, \text{ for all integers } n \geq 1.$$

Remarks

- One can use this recursion in order to compute a list of the first moments $M_1, M_2, M_3, \ldots$.
- From above one can express M_n with the help of the previous moments:

$$M_n = \frac{1-r}{2^n - 1} \sum_{j=0}^{n-1} \binom{n}{j} M_j.$$

The asymptotics of the moments M_n

We define the *exponential generating function*

$$M(z) = \sum_{n \geq 0} M_n \frac{z^n}{n!}.$$

We obtain

$$M(z) = r \cdot M(\tfrac{z}{2}) + (1-r) \cdot M(\tfrac{z}{2}) \cdot e^{\frac{z}{2}}.$$

(The above functional equation could also have been derived by using the self-similar properties of μ_r.)
The *Poisson transformed function* $\widehat{M}(z) = M(z) \cdot e^{-z}$ satisfies

$$\widehat{M}(z) = r \cdot \widehat{M}(\tfrac{z}{2}) \cdot e^{-\frac{z}{2}} + (1-r) \cdot \widehat{M}(\tfrac{z}{2}).$$

Herefrom, by iteration:

$$\widehat{M}(z) = \prod_{k\geq 1} \left(r \cdot e^{-\frac{z}{2^k}} + (1-r)\right).$$

As we are looking for the asymptotics of the moments M_n we are going to study the behaviour of $\widehat{M}(z)$ for $z \to \infty$.

This is based on the fact that $M_n \sim \widehat{M}(n)$.

Justification: by using *depoissonisation*.

The basic idea: extract the coefficients M_n from $M(z)$ using Cauchy's integral formula and the saddle point method.

The asymptotics of the moments M_n

This leads in our applications to an approximation

$$M_n = \widehat{M}(n)\big(1 + \mathcal{O}(\frac{1}{n})\big),$$

with more terms being available in principle.
We rewrite

$$\widehat{M}(z) = r \cdot \widehat{M}(\tfrac{z}{2}) \cdot e^{-\frac{z}{2}} + (1-r) \cdot \widehat{M}(\tfrac{z}{2}).$$

as

$$\widehat{M}(z) = (1-r) \cdot \widehat{M}(\tfrac{z}{2}) + R(z),$$

where $R(z) = r \cdot \widehat{M}(\frac{z}{2}) \cdot e^{-\frac{z}{2}}$ is considered to be an auxiliary function which we treat as a known function.

The asymptotics of the moments M_n

We compute the Mellin transform $\widehat{M}^*(s)$ of the function $\widehat{M}(z)$. We get

$$\widehat{M}^*(s) = (1-r)\cdot 2^s \cdot \widehat{M}^*(s) + R^*(s) = \frac{R^*(s)}{1-(1-r)\cdot 2^s}.$$

Now the function $\widehat{M}(z)$ can be obtained by applying the Mellin inversion formula, namely

$$\widehat{M}(z) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \widehat{M}^*(s)\cdot z^{-s} ds = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{R^*(s)}{1-(1-r)\cdot 2^s} \cdot z^{-s} ds,$$

where $0 < c < \log_2 \frac{1}{1-r}$.

The asymptotics of the moments M_n

We shift the integral to the right and take the residues with negative sign into account in order to estimate $\widehat{M}(z)$.
The function under the integral has simple poles at

$$s_k = \log_2 \frac{1}{1-r} + \frac{2k\pi \mathrm{i}}{\log 2},$$

$k \in \mathbb{Z}$. For these the residues with negative sign are

$$\frac{1}{\log 2} R^* \Big(\log_2 \frac{1}{1-r} + \frac{2k\pi \mathrm{i}}{\log 2} \Big) z^{-\log_2 \frac{1}{1-r} - \frac{2k\pi \mathrm{i}}{\log 2}},$$

with $R^*(s) = \int_0^\infty r \widehat{M}(\frac{z}{2}) \cdot e^{-\frac{z}{2}} \cdot z^{s-1} dz$.

For $k = 0$ the residue with negative sign is, using the definition of $R(z)$,

$$\frac{1}{\log 2} \cdot z^{\log_2(1-r)} \int_0^{\infty} r\widehat{M}(\frac{z}{2}) \cdot e^{-\frac{z}{2}} \cdot z^{\log_2 \frac{1}{1-r} - 1} dz.$$

This term plays an important role in the asymptotic behaviour of the nth moment M_n of the binomial distribution. In order to get this one collects all mentioned residues into a periodic function.

Theorem

The nth moment M_n of the binomial distribution μ_r admits the asymptotic estimate

$$M_n = \Phi(-\log_2 n) \cdot n^{\log_2(1-r)} \Big(1 + \mathcal{O}(\frac{1}{n})\Big),$$

for $n \longrightarrow \infty$,
where $\Phi(x)$ is a periodic function having period 1 and known Fourier coefficients. The mean (zeroth Fourier coefficient) of Φ is given by the expression

$$\frac{1}{\log 2} \int_0^\infty r\widehat{M}(\frac{z}{2}) \cdot e^{-\frac{z}{2}} \cdot z^{\log_2 \frac{1}{1-r} - 1} dz.$$

Remark. One can compute this integral numerically by taking for $\widehat{M}(\frac{z}{2})$ the first few terms of its Taylor expansion. These can be found from the recurrence for the numbers M_n.

The integral in the expresion of the zeroth Fourier coefficient can be written as

$$\int_0^\infty r\widehat{M}(\frac{z}{2}) \cdot e^{-\frac{z}{2}} \cdot z^{\log_2 \frac{1}{1-r}-1} dz = r\int_0^\infty e^{-z} \sum_{k\geq 0} M_k \frac{z^k}{2^k k!} z^{\log_2 \frac{1}{1-r}-1} dz$$

$$= r \sum_{k\geq 0} \frac{M_k}{2^k k!} \cdot \Gamma\Big(k + \log_2 \frac{1}{1-r}\Big).$$

This series is well suited for numerical computations. For example, let $r = 0.6$, then $M_{100} = 0.002453\ldots$ and the value predicted in the Theorem (without the oscillation) is $0.002491\ldots$.

Generalisation: the multinomial distribution

The Gray code distribution and its moments

The Gray code distribution. Definition

Definition

(Kobayashi) Let $I = I_{0,0} = [0,1]$ and

$$I_{n,j} = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \text{ for } j = 0,1,\dots,2^n-2, \qquad I_{n,2^n-1} = \left[\frac{2^n-1}{2^n}, 1\right],$$

for $n = 1,2,3,\dots$.

For each $0 < r < 1$ there exists a unique probability measure $\tilde{\mu}_r$ on I such that, for $j = 0,1,\dots,2^n-1$ and $n = 0,1,2,\dots$,

$$\tilde{\mu}_r(I_{n+1,2j}) = \begin{cases} r\tilde{\mu}_r(I_{n,j}) & j: \text{ even}, \\ (1-r)\tilde{\mu}_r(I_{n,j}) & j: \text{ odd}, \end{cases}$$

$$\tilde{\mu}_r(I_{n+1,2j+1}) = \begin{cases} (1-r)\tilde{\mu}_r(I_{n,j}) & j: \text{ even}, \\ r\tilde{\mu}_r(I_{n,j}) & j: \text{ odd}. \end{cases}$$

We call $\tilde{\mu}_r$ the *Gray code measure*.

The moments of the Gray code distribution

We have

$$M_n = \sum_{\omega \in \mathcal{W}} \tilde{\mu}_r(\omega) \cdot (val(\omega))^n$$

and

$$M_n = \lim_{m \to \infty} M_n^m,$$

where

$$M_n^m = \sum_{\omega \in \mathcal{W}_m} \tilde{\mu}_r(\omega) \cdot (val(\omega))^n.$$

We are looking for a recurrence relation between the moments of different orders.

Idea: study first the recursive behaviour of the moments of finite words M_n^m.

Reading a word $\omega \in \mathcal{W}_m$, $\omega = \omega_1\omega_2\ldots\omega_m$

- If $\omega_1 = 0$ then $val(\omega)$ lies in the left elementary interval of level 1.
  If $\omega_1 = 1$ then $val(\omega)$ lies in the right elementary interval of level 1.
- Reading ω_2 indicates for $val(\omega)$ the interval of level 2 inside the interval of level 1 indicated by ω_1.
  $\omega_2 = 0 \longrightarrow$ left interval, $\omega_2 = 1 \longrightarrow$ right interval.
- ω_k indicates the position of the interval of level k ($k \leq m$) that contains $val(\omega)$.

Remark. *A Markov chain model of the problem*
Let us now consider the Markov chain with state space $X = \{0, 1\} = D$ and transition probabilities

$$p(x, y) = \begin{cases} r, & x{=}y, \\ 1 - r, & x{\neq}y, \end{cases}$$

where $x, y \in \{0, 1\}$.
Generating finite random words ω (with respect to the distribution $\tilde{\mu}_r$) over the alphabet $D = \{0, 1\}$ is equivalent to the random walk described by the above Markov chain:
$X(k)$ indicates the k-th digit of ω, i.e., $X(k) = \omega_k$, $k \geq 1$.

We have, for $k = 1, 2, \ldots$:

$$\mathbb{P}(\omega_k = 0) = \tfrac{1}{2}\big((2r-1)^k + 1\big), \quad \mathbb{P}(\omega_k = 1) = \tfrac{1}{2}\big(1 - (2r-1)^k\big).$$

Some more definitions and notations

- For any $\omega = \omega_1\omega_2 \ldots \omega_m \in \mathcal{W}_m$,

$$\overline{\omega} := \overline{\omega}_1\overline{\omega}_2 \ldots \overline{\omega}_m \in \mathcal{W}_m, \text{ with } \overline{\omega}_k = 1 \oplus \omega_k, \quad k = 1, 2, \ldots, m.$$

  Analogously, for $\omega \in \mathcal{W}$.

- For any integer $k \geq 0$, $k = \sum_{j=1}^{m} \varepsilon_j(k) \cdot 2^j$, $\varepsilon_j \in \{0, 1\}$, $j = 1, 2 \ldots, m$ and $0 < r < 1$

$$\pi_{r,m}(k) := r^{m-\tilde{s}(k)} \cdot (1 - r)^{\tilde{s}(k)},$$

  $\tilde{s}(k)$ = the number of digits 1 in the Gray code $g(k)$ of k

  the *Gray digital sum* (Kobayashi, 2002).

- For any integer $m \geq 1$ and any word $\omega \in \mathcal{W}_m$ we define

$$\pi_r(\omega) := \pi_{r,m}(2^m \cdot val(\omega)).$$

Remarks.

- (Induction) For any positive integer m, any integer $0 \leq k \leq 2^m - 1$ and any $0 < r < 1$,

$$\tilde{\mu}_r(I_{m,k}) = \pi_{r,m}(k),$$

i.e., $\pi_{r,m}(k)$ is the probability that the starting block $\omega_1\omega_2\ldots\omega_m$ of a random word $\omega \in \mathcal{W}$ satisfies

$$\omega_j = \varepsilon_j(k) \in \{0,1\} \text{ for } j = 1, 2, \ldots, m, \text{ where } k = \sum_{j=1}^{m} \varepsilon_j(k) \cdot 2^j.$$

- With the above notations,

$$\tilde{\mu}_r(\omega_1\omega_2\ldots\omega_m) = \pi_r(\omega), \text{ for any } \omega = \omega_1\omega_2\ldots\omega_m \in \mathcal{W}_m.$$

The moments of the Gray code distribution

We have

$$M_n^m = \sum_{\omega \in \mathcal{W}_m} \pi_r(\omega) \cdot (val(\omega))^n$$
$$= \frac{r}{2^n} \sum_{\omega' \in \mathcal{W}_{m-1}} \pi_r(\omega') \cdot (val(\omega'))^n + \frac{1-r}{2^n} \sum_{\omega' \in \mathcal{W}_{m-1}} \pi_r(\omega') \cdot (1 + val(\overline{\omega'}))^n$$

Moments of the Gray code distribution

- Let ϕ be the bijection $\phi : \mathcal{W} \to \mathcal{W}$, $\phi(\omega) = \overline{\omega}$ and for any $m \geq 1$, ϕ_m the obvious (bijective) restriction $\phi_m : \mathcal{W}_m \to \mathcal{W}_m$.

  $\phi(\phi(\omega)) = \omega$, for all $\omega \in \mathcal{W}$,
  $\phi_m(\phi_m(\omega)) = \omega$, for all $\omega \in \mathcal{W}_m$, and $m \geq 1$.
- The moments $\overline{M}_n$ and $\overline{M}_n^m$ of the composed function $val \circ \phi$ with respect to the Gray code distribution:

$$\overline{M}_n^m = \sum_{\omega \in \mathcal{W}_m} \tilde{\mu}_r(\omega) \cdot (val(\phi(\omega)))^n = \sum_{\omega \in \mathcal{W}_m} \pi_r(\omega) \cdot (val(\overline{\omega}))^n,$$
$$\overline{M}_n = \sum_{\omega \in \mathcal{W}} \tilde{\mu}_r(\omega) \cdot (val(\phi(\omega)))^n = \sum_{\omega \in \mathcal{W}} \tilde{\mu}_r(\omega) \cdot (val(\overline{\omega}))^n.$$

The moments of the Gray code distribution

Theorem

The moments of the Gray distribution $\tilde{\mu}_r$ *satisfy the relations:*
$M_0 = \overline{M}_0 = 1$ *and*

$$M_n = \frac{r}{2^n} M_n + \frac{\overline{r}}{2^n} \sum_{j=0}^{n} \binom{n}{j} \overline{M}_j,$$

$$\overline{M}_n = \frac{\overline{r}}{2^n} \overline{M}_n + \frac{r}{2^n} \sum_{j=0}^{n} \binom{n}{j} M_j,$$

for all integers $n \geq 1$ *and* $\overline{r} = 1 - r$.

Remark. One can use these recursion relations in order to compute a list of the first few moments $M_1, M_2, \ldots,$ and $\overline{M}_1, \overline{M}_2, \ldots$.

The asymptotics of the moments M_n

The exponential generating functions

$$A(z) = \sum_{n \geq 0} M_n \frac{z^n}{n!}, \quad B(z) = \sum_{n \geq 0} \overline{M}_n \frac{z^n}{n!}.$$

From the above recursions we get

$$A(z) = r \cdot A(\frac{z}{2}) + \overline{r} \cdot e^{\frac{z}{2}} \cdot B(\frac{z}{2}), \qquad B(z) = \overline{r} \cdot B(\frac{z}{2}) + r \cdot e^{\frac{z}{2}} \cdot A(\frac{z}{2}),$$

and for the *Poisson transformed functions*,

$$\widehat{A}(z) = A(z) \cdot e^{-z} \text{ and } \widehat{B}(z) = B(z) \cdot e^{-z},$$

$$\widehat{A}(z) = \overline{r} \cdot \widehat{B}(\frac{z}{2}) + r \cdot e^{-\frac{z}{2}} \cdot \widehat{A}(\frac{z}{2}), \qquad \widehat{B}(z) = r \cdot \widehat{A}(\frac{z}{2}) + \overline{r} \cdot e^{-\frac{z}{2}} \cdot \widehat{B}(\frac{z}{2}).$$

The asymptotics of the moments M_n

Theorem

The nth moment M_n of the Gray code distribution $\tilde{\mu}_r$ admits the asymptotic estimate

$$M_n = \Phi(-\log_4 n) \cdot n^{\log_4(\bar{r}r)} \Big(1 + \mathcal{O}(\frac{1}{n})\Big),$$

for $n \to \infty$, where $\Phi(x)$ is a periodic function having period 1 and known Fourier coefficients. The mean (zeroth Fourier coefficient) of Φ is given by the expression

$$\frac{1}{\log 4} \int_0^\infty \Big(\bar{r}^2 \cdot e^{-\frac{z}{4}} \cdot \widehat{B}(\frac{z}{4}) + r \cdot e^{-\frac{z}{2}} \cdot \widehat{A}(\frac{z}{2})\Big) \cdot z^{\log_4 \frac{1}{\bar{r}r} - 1} dz.$$

For numerical computations, one cand use the equivalent expression

$$\frac{1}{\log 4}\left(\frac{\bar{r}^2}{\sqrt{\bar{r}r}}\sum_{k\geq 0}\frac{\overline{M}_k}{2^k k!}\cdot\Gamma\left(k+\log_4\frac{1}{\bar{r}r}\right)+r\cdot\sum_{k\geq 0}\frac{M_k}{2^k k!}\Gamma\left(k+\log_4\frac{1}{\bar{r}r}\right)\right)$$