Structural Properties of bounded Languages with Respect to Multiplication by a Constant

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Outline of the talk

- Abstract Numeration Systems
- Motivation - Main Question
- First Results
- Bounded Languages
- $\mathcal{B}_\ell$-Representation of an Integer
- Multiplication by $\lambda = \beta^\ell$

Definition (P. Lecomte, M. Rigo)

An *abstract numeration system* is a triple $S = (L, \Sigma, <)$ where L is a regular language over a totally ordered alphabet $(\Sigma, <)$.

Enumerating the words of L with respect to the genealogical ordering induced by $<$ gives a one-to-one correspondence

$$\mathrm{rep}_S : \mathbb{N} \to L \qquad \mathrm{val}_S = \mathrm{rep}_S^{-1} : L \to \mathbb{N}.$$

Example

$L = a^*, \ \Sigma = \{a\}$

n	0	1	2	3	4	$\cdots$
$\mathrm{rep}(n)$	ε	a	aa	aaa	$aaaa$	$\cdots$

Example

$L = \{a,b\}^*,\ \Sigma = \{a,b\},\ a < b$

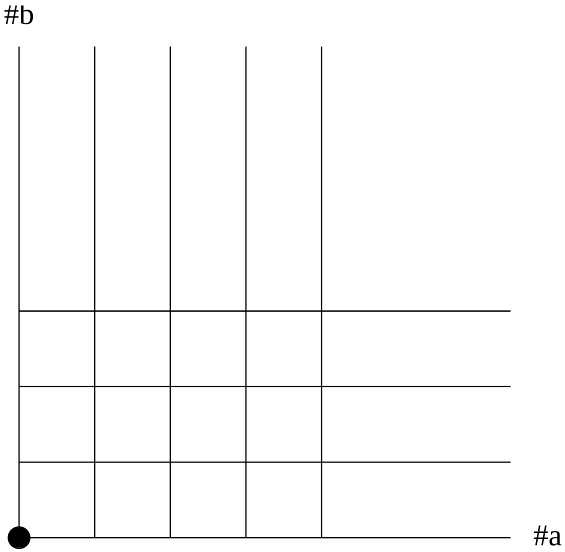
n	0	1	2	3	4	5	6	7	$\cdots$
$\mathrm{rep}(n)$	ε	a	b	aa	ab	ba	bb	aaa	$\cdots$

Example

$L = a^*b^*,\ \Sigma = \{a,b\},\ a < b$

n	0	1	2	3	4	5	6	$\cdots$
$\mathrm{rep}(n)$	ε	a	b	aa	ab	bb	aaa	$\cdots$

$$\mathrm{val}(a^p b^q) = \frac{1}{2}(p+q)(p+q+1) + q$$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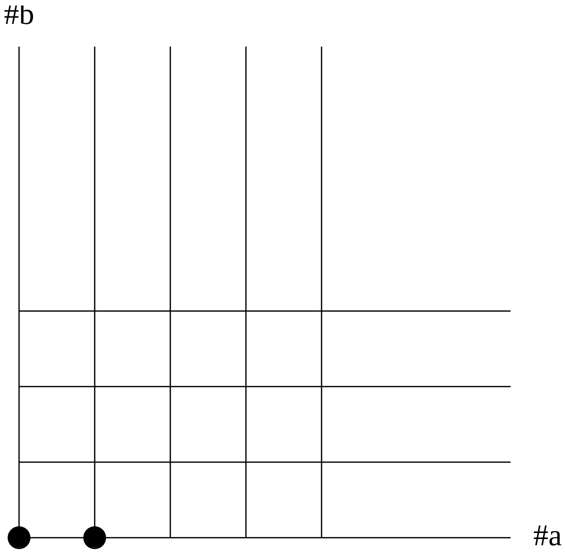
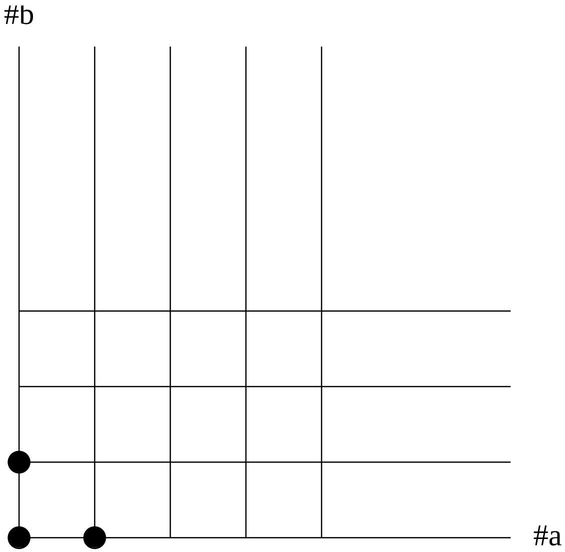
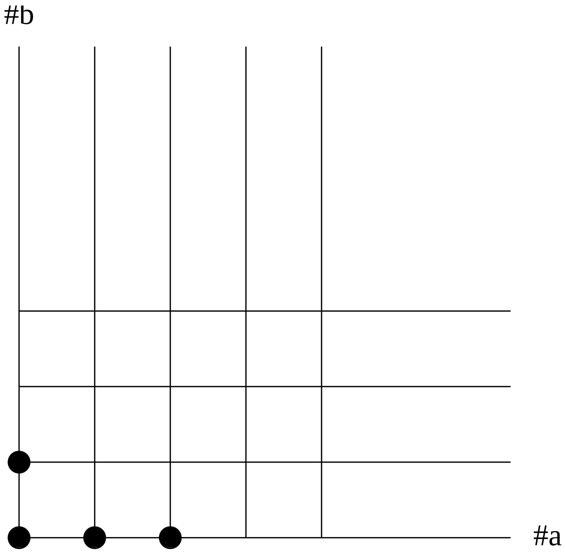
Abstract Numeration Systems

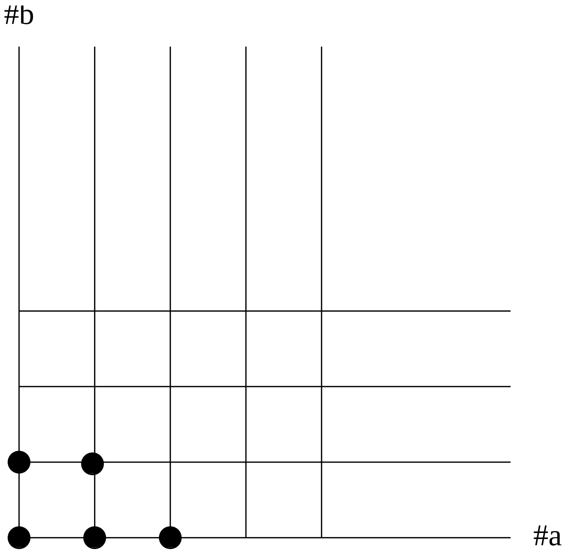


Abstract Numeration Systems



Abstract Numeration Systems

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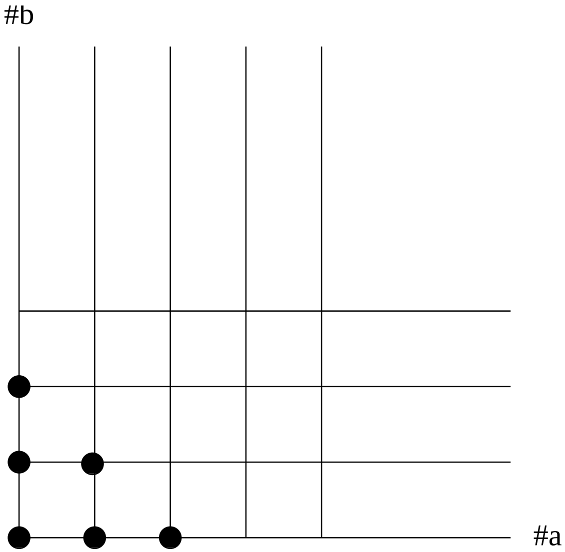
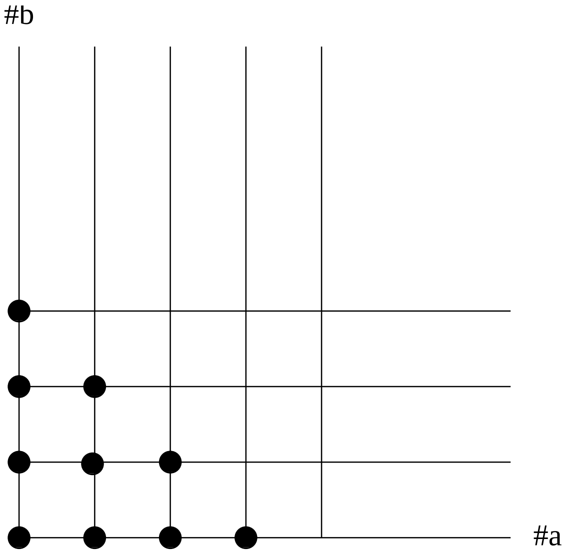
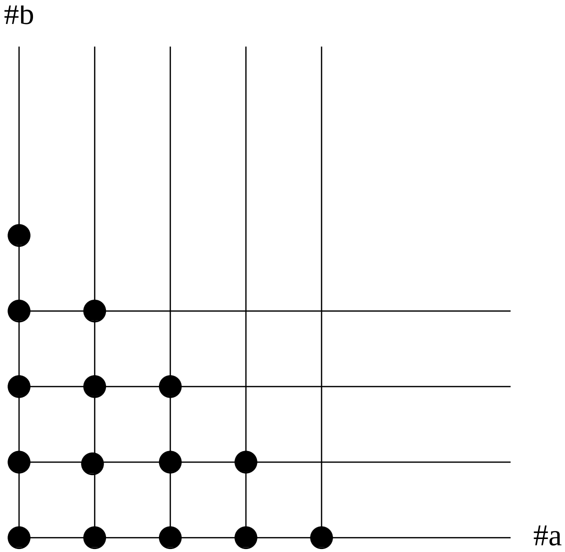


Abstract Numeration Systems

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Remark

This generalizes "classical" Pisot systems like integer base systems or Fibonacci system.

$$L = \{\varepsilon\} \cup \{1, \dots, k-1\}\{0, \dots, k-1\}^* \text{ or } L = \{\varepsilon\} \cup 1\{0, 01\}^*$$

Definition

A set $X \subseteq \mathbb{N}$ is *S-recognizable if* $\mathrm{rep}_S(X) \subseteq \Sigma^*$ *is a regular language (accepted by a DFA).*

How to compute in such a numeration system ?

More precisely, how act arithmetic operations like addition, multiplication by a constant, ...?

$\longrightarrow$ We focus on multiplication by a constant.

Question : Multiplication by a Constant

If $S = (L, \Sigma, <)$ is an abstract numeration system, can we find some necessary and sufficient condition on $\lambda \in \mathbb{N}$ such that for any S-recognizable set X, the set λX is still S-recognizable ?

$$X \ S\text{-rec} \quad \overset{?}{\longrightarrow} \quad \lambda X \ S\text{-rec}$$

Theorem (Translation, P. Lecomte, M. Rigo)

Let $S = (L, \Sigma, <)$ *be an abstract numeration system and* $X \subseteq \mathbb{N}$*. For each* $t \in \mathbb{N}$*,* $X + t$ *is* S*-recognizable if and only if* X *is* S*-recognizable.*

Definition

We denote by $\mathbf{u}_L(n)$ the number of words of length n belonging to L.

Theorem (Polynomial Case, M. Rigo)

Let $L \subseteq \Sigma^*$ *be a regular language such that* $\mathbf{u}_L(n)$ *is* $\Theta(n^k)$ *for some* $k \in \mathbb{N}$ *and* $S = (L, \Sigma, <)$*. Preservation of* S*-recognizability after multiplication by* λ *holds only if* $\lambda = \beta^{k+1}$ *for some* $\beta \in \mathbb{N}$*.*

Definition

A language L is *slender* if $\mathbf{u}_L(n) \in O(1)$.

Theorem (Slender Case, E. C., M. Rigo)

Let $L \subset \Sigma^*$ be a slender regular language and $S = (L, \Sigma, <)$. A set $X \subseteq \mathbb{N}$ is S-recognizable if and only if X is a finite union of arithmetic progressions.

Corollary

Let S be a numeration system built on a slender language. If $X \subseteq \mathbb{N}$ is S-recognizable then λX is S-recognizable for all $\lambda \in \mathbb{N}$.

Theorem (P. Lecomte, M. Rigo)
Let $\beta \in \mathbb{N} \setminus \{0\}$. For the abstract numeration system

$$S = (a^* b^*, \{a, b\}, a < b),$$

multiplication by β^2 preserves S-recognizability if and only if β is an odd integer.

$\longrightarrow$ We focus on abstract numeration systems built on bounded languages.

Notation

We denote by $\mathcal{B}_\ell = a_1^* \cdots a_\ell^*$ the bounded language over the totally ordered alphabet $\Sigma_\ell = \{a_1 < \ldots < a_\ell\}$ of size $\ell \geq 1$.

We consider abstract numeration systems of the form $(\mathcal{B}_\ell, \Sigma_\ell)$ and we denote by rep_ℓ and val_ℓ the corresponding bijections.

A set $X \subseteq \mathbb{N}$ is said to be *$\mathcal{B}_\ell$-recognizable* if $\mathrm{rep}_\ell(X)$ is a regular language over the alphabet Σ_ℓ.

If w is a word over Σ_ℓ, $|w|$ denotes its length and $|w|_{a_j}$ counts the number of letters a_j's appearing in w. The *Parikh mapping* Ψ maps a word $w \in \Sigma_\ell^*$ onto the vector $\Psi(w) := (|w|_{a_1}, \ldots, |w|_{a_\ell})$.

In this context, multiplication by a constant λ can be viewed as a transformation

$$f_\lambda : \mathcal{B}_\ell \to \mathcal{B}_\ell.$$

The question becomes then :
Can we determine some necessary and sufficient condition under which this transformation preserves regular subsets of $\mathcal{B}_\ell$ *?*

Example

Let $\ell = 2$, $\Sigma_2 = \{a, b\}$ and $\lambda = 25$.

$$\begin{array}{ccc} 8 & \xrightarrow{\times 25} & 200 \\ \mathrm{rep}_2 \downarrow & & \downarrow \mathrm{rep}_2 \\ a\,b^2 & \xrightarrow{f_{25}} & a^9\,b^{10} \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xrightarrow{\times \lambda} & \mathbb{N} \\ \mathrm{rep}_\ell \downarrow & & \downarrow \mathrm{rep}_\ell \\ \mathcal{B}_\ell & \xrightarrow{f_\lambda} & \mathcal{B}_\ell \end{array}$$

Thus multiplication by $\lambda = 25$ induces a mapping f_λ onto $\mathcal{B}_2$ such that for $w, w' \in \mathcal{B}_2$, $f_\lambda(w) = w'$ if and only if $\mathrm{val}_2(w') = 25\,\mathrm{val}_2(w)$.

$\mathcal{B}_\ell$-Representation of an Integer

We set

$$\mathbf{u}_\ell(n) := \mathbf{u}_{\mathcal{B}_\ell}(n) = \#(\mathcal{B}_\ell \cap \Sigma_\ell^n) \quad \text{and} \quad \mathbf{v}_\ell(n) := \#(\mathcal{B}_\ell \cap \Sigma_\ell^{\leq n}) = \sum_{i=0}^{n} \mathbf{u}_\ell(i).$$

Lemma

For all integers $\ell \geq 1$ and $n \geq 0$, we have

$$\mathbf{u}_{\ell+1}(n) = \mathbf{v}_\ell(n) \quad \textit{and} \quad \mathbf{u}_\ell(n) = \binom{n+\ell-1}{\ell-1}.$$

Lemma

Let $\ell \in \mathbb{N} \setminus \{0\}$ and $S = (a_1^* \cdots a_\ell^*, \{a_1 < \cdots < a_\ell\})$. We have

$$\mathrm{val}_\ell(a_1^{n_1} \cdots a_\ell^{n_\ell}) = \sum_{i=1}^{\ell} \binom{n_i + \cdots + n_\ell + \ell - i}{\ell - i + 1}.$$

Corollary (Lehmer 1964, Katona 1966, Fraenkel 1982)

Let $\ell \in \mathbb{N} \setminus \{0\}$. Any positive integer n can be uniquely written as

$$n = \binom{z_\ell}{\ell} + \binom{z_{\ell-1}}{\ell - 1} + \cdots + \binom{z_1}{1} \tag{1}$$

with $z_\ell > z_{\ell-1} > \cdots > z_1 \geq 0$.

Example

Consider the words of length 3 in the language $a^*b^*c^*$,

$$aaa < aab < aac < abb < abc < acc < bbb < bbc < bcc < ccc.$$

We have $\mathrm{val}_3(aaa) = \binom{5}{3} = 10$ and $\mathrm{val}_3(acc) = 15$. If we apply the erasing morphism $\varphi : \{a, b, c\} \to \{a, b, c\}^*$ defined by

$$\varphi(a) = \varepsilon, \varphi(b) = b, \varphi(c) = c$$

on the words of length 3, we get

$$\varepsilon < b < c < bb < bc < cc < bbb < bbc < bcc < ccc.$$

So we have $\mathrm{val}_3(acc) = \mathrm{val}_3(aaa) + \mathrm{val}_2(cc)$ where val_2 is considered as a map defined on the language b^*c^*.

Algorithm computing $\mathrm{rep}_\ell(n)$.

Let `n` be an integer and `l` be a positive integer.

```
For i=l,l-1,...,1 do
if n>0,
find t such that (t choose i) ≤ n < (t+1 choose i)
z(i)←t
n←n-(t choose i)
otherwise, z(i)←i-1
```

Consider now the triangular system having $\alpha_1, \ldots, \alpha_\ell$ as unknowns

$$\alpha_i + \cdots + \alpha_\ell = z(\ell - i + 1) - \ell + i, \quad i = 1, \ldots, \ell.$$

One has $\mathrm{rep}_\ell(\mathtt{n}) = a_1^{\alpha_1} \cdots a_\ell^{\alpha_\ell}$.

Multiplication by $\lambda = \beta^\ell$

Remark

We have $\mathbf{u}_{\mathcal{B}_\ell}(n) \in \Theta(n^{\ell-1})$.

So we have to focus only on multiplicators of the kind

$$\lambda = \beta^\ell.$$

Lemma

Let ℓ, $\beta \in \mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}$ large enough, we have

$$|\mathrm{rep}_\ell(\beta^\ell n)| = \beta\, |\mathrm{rep}_\ell(n)| + \left\lceil \frac{(\beta-1)(\ell+1)}{2} \right\rceil - i$$

with $i \in \{0, 1, \ldots, \beta\}$.

Multiplication by $\lambda = \beta^\ell$

Lemma

Let $\ell, \beta \in \mathbb{N} \setminus \{0\}$. Define $c_\ell, c_{\ell-1}, \ldots, c_1$ recursively by

$$c_{k+1} = k!\,(\beta^{\ell-k} - 1) \sum_{i=k}^{\ell} \frac{S_1(i,k)}{i!} - \sum_{i=k+2}^{\ell} \sum_{j=k+1}^{i} \frac{S_1(i,j)\,j!}{i!\,(j-k)!} c_i^{j-k}$$

where $S_1(i,j)$ are the unsigned Stirling numbers of the first kind. Then we have

$$\beta^\ell \left(\binom{q+\ell}{\ell} + \binom{q+\ell-1}{\ell-1} + \cdots + \binom{q}{1} \right)$$
$$= \binom{\beta q + c_\ell + \ell - 1}{\ell} + \binom{\beta q + c_{\ell-1} + \ell - 2}{\ell - 1} + \cdots + \binom{\beta q + c_1}{1},$$

for all $q \in \mathbb{R}$.

Remark

If all c_k, $1 \le k \le \ell$, are integers and $c_\ell \ge c_{\ell-1} \ge \cdots \ge c_1$, then

$$\mathrm{rep}_\ell(\beta^\ell \mathrm{val}_\ell(a_\ell^q)) = a_1^{c_\ell - c_{\ell-1}} a_2^{c_{\ell-1} - c_{\ell-2}} \cdots a_{\ell-1}^{c_2 - c_1} a_\ell^{\beta q + c_1}$$

for all $q \ge -c_1/\beta$, hence $f_{\beta^\ell}(a_\ell^*)$ is regular.

Explicit forms for c_ℓ and $c_{\ell-1}$:

$$c_\ell = \frac{(\beta-1)(\ell+1)}{2} \quad \text{for } \ell \ge 2,$$

$$c_{\ell-1} = \frac{(\beta-1)(\ell+1)}{2} - \frac{(\beta^2-1)(\ell+1)}{24} \quad \text{for } \ell \ge 3.$$

Lemma

Let A be a k-dimensional linear subset of $\mathbb{N}^\ell$ for some integer $1 \le k < \ell$ and $B = \Psi^{-1}(A) \cap \mathcal{B}_\ell$ be the corresponding subset of $\mathcal{B}_\ell$. If $\Psi(f_{\beta^\ell}(B))$ contains a sequence $x^{(n)} = (x_1^{(n)}, \dots, x_\ell^{(n)})$ such that $\min(x_{j_1}^{(n)}, \dots, x_{j_{k+1}}^{(n)}) \to \infty$ as $n \to \infty$ for some $j_1 < \cdots < j_{k+1}$, then $f_{\beta^\ell}(B)$ is not regular.

Proposition

If $c_\ell \notin \mathbb{Z}$ or $c_{\ell-1} \notin \mathbb{Z}$ with $\ell \ge 3$, then $f_{\beta^\ell}(a_\ell^*)$ is not regular.

Proposition

If $c_\ell, c_{\ell-1} \in \mathbb{Z}$ with $\ell \ge 3$, $\beta \ge 2$, then $f_{\beta^\ell}(a_1^* a_\ell^*)$ is not regular.

Theorem (E. C., M. Rigo, W. Steiner)

Let $\ell,\, \beta \in \mathbb{N} \setminus \{0\}$. For the abstract numeration system

$$S = (a_1^* \dots a_\ell^*, \{a_1 < \dots < a_\ell\}),$$

multiplication by β^ℓ preserves S-recognizability if and only if one of the following condition is satisfied :

- $\ell = 1$
- $\beta = 1$
- $\ell = 2$ *and* β *is an odd integer.*