On relationships between shift radix systems and canonical number systems

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Graz, April 16, 2007

(partially supported by the Austrian Science Foundation
(FWF), Project S9610)

Shift radix systems

Let $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ $(d \geq 1)$ and

$$\tau_{\mathbf{r}} \; : \; \begin{cases} \mathbb{Z}^d \to \mathbb{Z}^d \\ (a_1, \dots, a_d) \longmapsto (a_2, \dots, a_d, -\lfloor r_1 a_1 + \cdots + r_d a_d \rfloor) \end{cases}$$

The mapping $\tau_{\mathbf{r}}$ is called a *shift radix system* (SRS) if for all $\mathbf{a} \in \mathbb{Z}^d$ we can find some $n \in \mathbb{N}$ with $\tau_{\mathbf{r}}^n(\mathbf{a}) = (0, \dots, 0)$.

We are interested in the following sets:

$$\mathcal{D}_d^0 \;=\; \left\{ \mathbf{r} \in \mathbb{R}^d \; : \; \tau_{\mathbf{r}} \text{ is a shift radix system} \right\} \quad \text{and}$$

$$\mathcal{D}_d \;=\; \left\{ \mathbf{r} \in \mathbb{R}^d \; : \; (\tau_{\mathbf{r}}^k(\mathbf{a}))_{k \geq 0} \text{ is ultimately periodic for all } \mathbf{a} \in \mathbb{Z}^d \right\}.$$

Relationships of $\mathcal{D}_d$ and $\mathcal{D}_d^0$

$\mathcal{D}_d$ is related to the *Schur-Cohn region*:

$$\mathcal{E}_d = \{(r_1, \ldots, r_d) \in \mathbb{R}^d \,:\, X^d + r_d X^{d-1} + \cdots + r_2 X + r_1 \text{ has only roots } y \in \mathbb{C} \text{ with } |y| < 1\}$$

- $\mathcal{E}_d \subseteq \mathcal{D}_d \subseteq \overline{\mathcal{E}_d}$
- $\mathrm{int}\,(\mathcal{D}_d) = \mathcal{E}_d$

$\mathcal{D}_d^0$ is related to *number systems*:

- β-expansions with finiteness property (F)
- canonical number systems

Shift radix systems and β-expansions

Theorem (Hollander (1996))

Let $d > 1$ and $\beta > 1$ be a Pisot number with minimal polynomial $X^d - b_1X^{d-1} - \cdots - b_{d-1}X - b_d$. Set

$$r_j \ := \ b_j\beta^{-1} + b_{j+1}\beta^{-2} + \cdots + b_d\beta^{j-d-1} \quad (2 \leq j \leq d).$$

(Note: $X^d - b_1X^{d-1} - b_2X^{d-2} - \cdots - b_d$

$$= (X - \beta)(X^{d-1} + r_2X^{d-2} + \cdots + r_d))$$

Then $(r_d, \dots, r_2) \in \mathcal{D}_{d-1}^0$ if and only if

$$\mathbb{Z}\left[\frac{1}{\beta}\right] \cap [0, \infty)$$

coincides with the set of nonnegative real numbers having finite greedy expansion with respect to β.

Canonical number systems

Let $P = p_d X^d + \cdots + p_0 \in \mathbb{Z}[X]$ with $p_0 \neq 0$, $p_d = 1$, and define $T_P \; : \; \mathbb{Z}[X] \longrightarrow \mathbb{Z}[X]$ by

$$T_P\Big(\sum_{i=0}^{m} a_i X^i\Big) = \sum_{i=0}^{m-1} a_{i+1} X^i \; - \; \lfloor \frac{a_0}{p_0} \rfloor \sum_{i=0}^{d-1} p_{i+1} X^i.$$

Definition (Kátai , Szabó (1975), Kátai , Kovács (1980), Kovács (1981), Gilbert (1981), Pethő (1991))

P is called a *CNS polynomial* if for each $A \in \mathbb{Z}[X]$ there is a $k \in \mathbb{N}$ such that $T_P^k(A) = 0$. In this case the pair $(\alpha, \{0, \ldots, |P(0)| - 1\})$ is called a *canonical number system (CNS)* where α is a root of P.

Set

$$\mathcal{C}_d^0 = \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d : X^d + p_{d-1} X^{d-1} + \cdots + p_0 \text{ CNS polynomial}\}$$

$$\mathcal{C}_d = \{(p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d \; : \; p_0 \neq 0 \text{ and } T_{X^d + p_{d-1} X^{d-1} + \cdots + p_0} \text{ has only finite orbits}\}.$$

Shift radix systems and canonical number systems

We have the following relations for $p_0, \dots, p_{d-1} \in \mathbb{Z}, p_0 \neq 0$ (Akiyama, Borbély, B., Pethő, Thuswaldner (2005)):

- $(p_0, p_1, \dots, p_{d-1}) \in \mathcal{C}_d^0$ if and only if

$$(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}_d^0$$

- $(p_0, p_1, \dots, p_{d-1}) \in \mathcal{C}_d$ if and only if

$$(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}) \in \mathcal{D}_d$$

Structure of the sets $\mathcal{C}_d^0$, $\mathcal{C}_d$, $\mathcal{D}_d^0$ and $\mathcal{D}_d$

$$\mathcal{C}_1^0 = \{p_0 \in \mathbb{Z} : p_0 \geq 2\} \qquad \text{(Grünwald (1885))}$$

$$\mathcal{C}_1 = \{p_0 \in \mathbb{Z} : |p_0| \geq 1\}, \quad \mathcal{D}_1 = [-1, 1], \quad \mathcal{D}_1^0 = [0, 1)$$

(Akiyama, Borbély, B., Pethő, Thuswaldner (2005))

$$\mathcal{C}_2^0 = \{(p_0, p_1) \in \mathbb{Z}^2 : -1 \leq p_1 \leq p_0 \geq 2\}$$

(Kátai, Szabó (1975), Kátai, Kovács (1981), Gilbert (1981), Grossman (1985), ...)

$$\mathcal{C}_2 = \{(p_0, p_1) \in \mathbb{Z}^2 : -p_0 \leq p_1 \leq p_0 + 1, p_0 \geq 2\}$$

Structure of the sets $\mathcal{C}_d^0$, $\mathcal{C}_d$, $\mathcal{D}_d^0$ and $\mathcal{D}_d$

Partial results for the sets $\mathcal{D}_d, \mathcal{D}_d^0 \quad (d \geq 2)$ and $\mathcal{C}_d, \mathcal{C}_d^0 \quad (d \geq 3)$ are known:

$d = 2$: Gilbert (1981), Akiyama et al. (2006), Surer (2006), ...

$d = 3$: Scheicher, Thuswaldner (2004), Akiyama et al. (2006), ...

$d \geq 3$: Kovács (1981), Kovács, Pethő (1983, 1991), Akiyama, Pethő (2002), Scheicher, Thuswaldner (2004), Pethő (2004), Akiyama, Rao (2004), Akiyama et al. (2004, 2006), ...

But:

$$\mathcal{D}_2^0 \quad = \quad ?, \qquad \mathcal{D}_2 \quad = \quad ?$$

Structure of the sets $\mathcal{C}_2^0$, $\mathcal{C}_2$, $\mathcal{D}_1^0$ and $\mathcal{D}_1$

$$\mathcal{C}_2 = \{(p_0, p_1) \in \mathbb{Z}^2 \,:\, -p_0 \le p_1 \le p_0 + 1, p_0 \ge 2\}, \qquad \mathcal{D}_1 = [-1, 1]$$

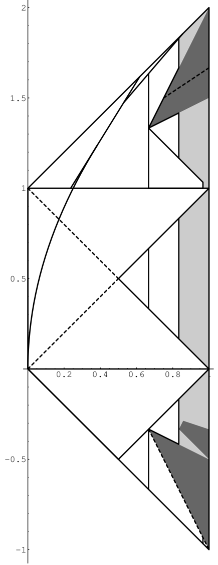
hence

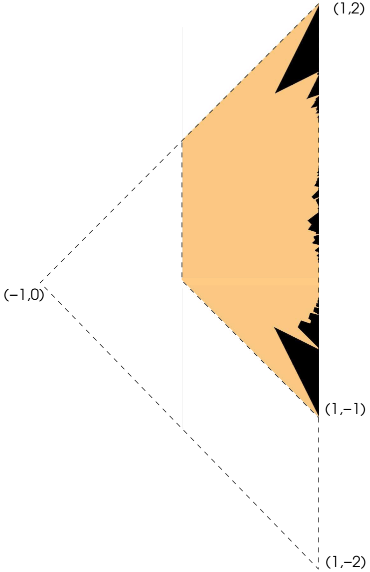
$$\left\{ \frac{p_1}{p_0} \,:\, (p_0, p_1) \in \mathcal{C}_2 \right\} \text{ "approximates" } [-1, 1] = \overline{\mathcal{D}_1} \quad \text{for } p_0 \to \infty.$$

$$\mathcal{C}_2^0 = \{(p_0, p_1) \in \mathbb{Z}^2 \,:\, -1 \le p_1 \le p_0 \ge 2\}, \qquad \mathcal{D}_1^0 = [0, 1)$$

hence

$$\left\{ \frac{p_1}{p_0} \,:\, (p_0, p_1) \in {\mathcal{C}_2}^0 \right\} \text{ "approximates" } [0, 1] = \overline{{\mathcal{D}_1}^0} \quad \text{for } p_0 \to \infty.$$





Approximations of $\mathcal{D}_2{}^0$ and of $\left\{ \left(\frac{p_2}{p_0}, \frac{p_1}{p_0} \right) : (p_0, p_1, p_2) \in \mathcal{C}_3{}^0 \right\}$

Approximation of SRS by CNS

In the following let $d \geq 2$.

For the approximation we use

- suitable sets: For $M \in \mathbb{N}_{>0}$ let

$$\mathcal{C}_d^0(M) = \left\{ \left(\frac{p_{d-1}}{M}, \dots, \frac{p_1}{M} \right) \in \mathbb{R}^{d-1} \; : \; (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d^0 \right\}$$

  and

$$\mathcal{C}_d(M) = \left\{ \left(\frac{p_{d-1}}{M}, \dots, \frac{p_1}{M} \right) \in \mathbb{R}^{d-1} \; : \; (M, p_1, \dots, p_{d-1}) \in \mathcal{C}_d \right\}.$$

- a notion of limit of sets:

Convergence of sets

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a topological space Z.

- A point $z \in Z$ belongs to the *(topological) lower limit* $\underline{\mathrm{Lim}}_{n\to\infty} A_n$ if every neighborhood of z intersects all A_n for n sufficiently large.
- A point $z \in Z$ belongs to the *(topological) upper limit* $\overline{\mathrm{Lim}}_{n\to\infty} A_n$ if every neighborhood of z intersects A_n for infinitely many n.
- The set A is said to be the *(topological) limit* of $(A_n)_{n\in\mathbb{N}}$ if $A = \underline{\mathrm{Lim}}_{n\to\infty} A_n = \overline{\mathrm{Lim}}_{n\to\infty} A_n$. We write

$$A = \mathop{\mathrm{Lim}}_{n\to\infty} A_n$$

Analogously: $\mathrm{Lim}_{x\to x_0} A_x$ for $x_0 \in \mathbb{R}, I \subseteq \mathbb{R}$ and $(A_x)_{x\in I}$ a collection of subsets of a topological space.

Approximation of the closure of $\mathcal{D}_d$

Theorem

$$\operatorname*{Lim}_{M\to\infty} \mathcal{C}_d(M) = \overline{\mathcal{D}_{d-1}}$$

For $x \in \mathbb{R}$ we need the following “cuts” of $\mathcal{D}_d$ and $\mathcal{D}_d^0$:

$$\begin{aligned}
\mathcal{D}_d(x) &= \left\{(r_2,\dots,r_d) \in \mathbb{R}^{d-1} : (x, r_2, \dots, r_d) \in \mathcal{D}_d\right\},\\
\mathcal{D}_d^0(x) &= \left\{(r_2,\dots,r_d) \in \mathbb{R}^{d-1} : (x, r_2, \dots, r_d) \in \mathcal{D}_d^0\right\}.
\end{aligned}$$

Theorem

$$\operatorname*{Lim}_{x\to 0} \mathcal{D}_d(x) = \overline{\mathcal{D}_{d-1}}$$

Lebesgue measure of $\mathcal{D}_d^0$ and $\mathcal{D}_d$

Theorem (Akiyama, Borbély, B., Pethő, Thuswaldner (2005))

$\mathcal{D}_d$ and $\mathcal{D}_d^0$ are Lebesgue measurable, and $\lambda_d(\mathcal{D}_d) = \lambda_d(\mathcal{E}_d)$.

Theorem

(i) $\lim_{x\to 0} \lambda_{d-1}\left(\mathcal{D}_d(x) \triangle \mathcal{D}_{d-1}\right) = 0.$

(ii) $\lim_{x\to 0} \lambda_{d-1}\left(\mathcal{D}_d^0(x) \triangle \mathcal{D}_{d-1}^0\right) = 0.$

(iii) *For $M \in \mathbb{N}_{>0}$ set*

$$\mathcal{W}_d^0(M) = \bigcup_{\mathbf{x}\in\mathcal{C}_d^0(M)} \left\{\mathbf{x}' \in \mathbb{R}^{d-1} : ||\mathbf{x}' - \mathbf{x}||_\infty \leq \frac{1}{2M}\right\}.$$

Then we have

$$\lim_{M\to\infty} \lambda_{d-1}(\mathcal{W}_d^0(M) \triangle \overline{\mathcal{D}_{d-1}^0}) = 0.$$

Lebesgue measure of $\mathcal{D}_d^0$ and $\mathcal{D}_d$

For $M \in \mathbb{N}_{>0}$ we set

$$N^0(d,M) = |\{(p_1,\dots,p_{d-1}) \in \mathbb{Z}^{d-1} : (M,p_1,\dots,p_{d-1}) \in \mathcal{C}_d^0\}|,$$
$$N(d,M) = |\{(p_1,\dots,p_{d-1}) \in \mathbb{Z}^{d-1} : (M,p_1,\dots,p_{d-1}) \in \mathcal{C}_d\}|.$$

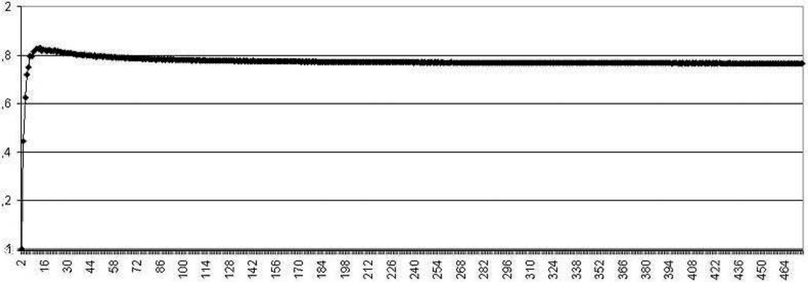
We are interested in the frequencies for $\mathcal{C}_d^0(M)$ and $\mathcal{C}_d(M)$

$$\frac{N^0(d,M)}{M^{d-1}} \qquad \text{and} \qquad \frac{N(d,M)}{M^{d-1}}$$

for $M \to \infty$.

Remark
$N^0(2,M) = |\{p_1 \in \mathbb{Z} : X^2 + p_1X + M \text{ CNS polynomial}\}| = M+2$, hence

$$\lim_{M\to\infty} \frac{N^0(2,M)}{M} = 1 = \lambda_1([0,1)) = \lambda_1(\mathcal{D}_1^0).$$



The behavior of $N^0(3, M)/M^2$ for $2 \leq M \leq 464$.

Apparently $N^0(3, M)/M^2 \to 1.766\ldots \quad \simeq \lambda_2(\mathcal{D}_2^0)$

Lebesgue measure of $\mathcal{D}_d^0$ and $\mathcal{D}_d$

Theorem

(i)

$$\lim_{M\to\infty} \frac{N^0(d,M)}{M^{d-1}} = \lambda_{d-1}\left(\mathcal{D}_{d-1}^0\right)$$

(ii)

$$\lim_{M\to\infty} \frac{N(d,M)}{M^{d-1}} = \lambda_{d-1}\left(\mathcal{D}_{d-1}\right)$$

Shift radix systems and canonical number systems

Open questions:

- Is it true that

$$\operatorname*{Lim}_{x\to 0} \mathcal{D}_d^0(x) = \overline{\mathcal{D}_{d-1}^0} \quad ?$$

- Can we estimate the number of Pisot numbers of a given trace having property (F) by shift radix systems?