Multidimensional continued fractions, numeration and discrete geometry

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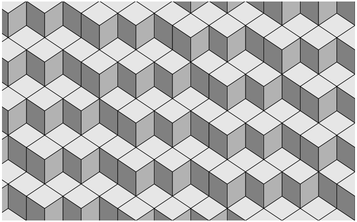
Recognition of arithmetic discrete planes

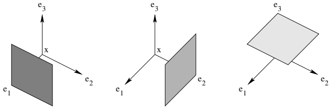
Problem

Given a set of points in $\mathbb{Z}^d$, does there exist a standard arithmetic discrete plane that contains them?



Arithmetic discrete planes

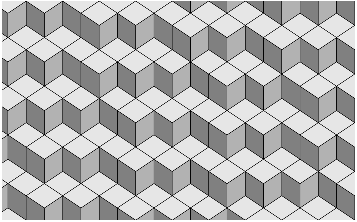
Let $(\vec{x}, 1^*)$, $(\vec{x}, 2^*)$, $(\vec{x}, 3^*)$ be the following faces:



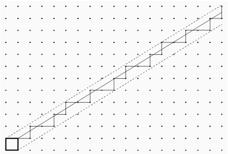
Definition

A standard arithmetic discrete plane or stepped plane is defined as

$$\mathcal{P}_{\vec{\alpha},\rho} = \{(\vec{x}, i^*) \mid 0 \leq \langle \vec{x}, \vec{\alpha} \rangle + \rho < \langle \vec{e}_i, \vec{\alpha} \rangle\}.$$



Discrete lines and Sturmian words



One can code a standard arithmetic discrete line (Freeman code) over the two-letter alphabet $\{0, 1\}$. One gets a Stumian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$

0100101001001010010100100101

Question

How to recognize that a finite word is Sturmian?

Finite Sturmian words

$$0110110101101101$$

We consider the substitutions

$$\sigma_0 : 0 \mapsto 0,\ \ \sigma_0 : 1 \mapsto 10$$

$$\sigma_1 : 0 \mapsto 01,\ \ \sigma_1 : 1 \mapsto 1$$

One has

$$01\,1\,01\,1\,01\,01\,1\,01\,1\,01 = \sigma_1(0101001010)$$

$$0\,10\,10\,0\,10\,10 = \sigma_0(011011)$$

$$01\,1\,01\,1 = \sigma_1(0101)$$

$$01\,01 = \sigma_1(00)$$

$\rightsquigarrow$ Continued fractions and Ostrowski numeration system

A classical recognition problem in discrete geometry

in connection with

- Geometric representations, Rauzy fractals, tilings
- Multidimensional continued fractions and S-adic systems

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We need a multidimensional notion of

- Substitutions

  $\rightsquigarrow$ Arnoux-Ito-Ei's formalism for unimodular morphisms of the free group

- Continued fraction algorithm

  $\rightsquigarrow$ Brun's algorithm

- Words

  $\rightsquigarrow$ Stepped surfaces

Substitution

Let σ be a substitution over $\mathcal{A}$.

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Example:

$$\sigma(1) = 12,\ \sigma(2) = 13,\ \sigma(3) = 1.$$

Substitution

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The incidence matrix $\mathbf{M}_\sigma$ of σ is the matrix defined by:

$$\mathbf{M}_\sigma = (|\sigma(j)|_i)_{(i,j)\in\mathcal{A}^2} ,$$

where $|\sigma(j)|_i$ is the number of occurrences of the letter i in $\sigma(j)$.

Substitution

Let σ be a substitution over $\mathcal{A}$.

Unimodular substitution

A substitution σ is unimodular if $\det \mathbf{M}_\sigma = \pm 1$.

Abelianization

Let d stand for the cardinality of $\mathcal{A}$. Let $\vec{l} : \mathcal{A}^\star \to \mathbb{N}^d$ be the Parikh mapping:

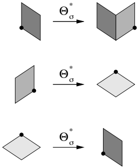
$$\vec{l}(w) = {}^t(|w|_1, |w|_2, \cdots, |w|_d).$$

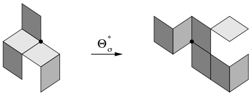
Generalized substitution

Generalized substitution [Arnoux-Ito]

Let σ be a unimodular substitution. We call generalized substitution the following tranformation acting on the faces $(\vec{x}, i^*)$ defined by:

$$\Theta_\sigma^*(\vec{x}, i^*) = \bigcup_{k \in \mathcal{A}} \bigcup_{P,\, \sigma(k) = PiS} \left(\mathbf{M}_\sigma^{-1}\left(\vec{x} - \vec{l}(P)\right), k^*\right).$$





Stepped surface

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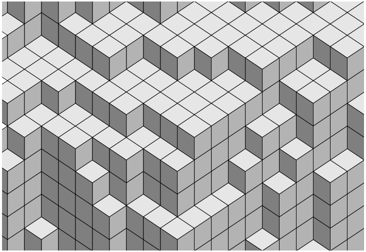

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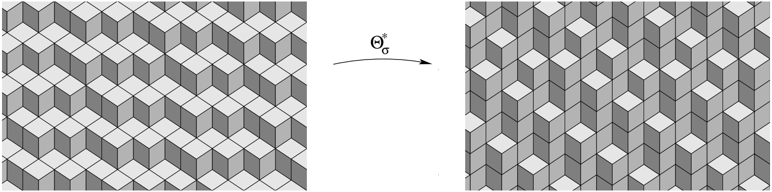
Recognition [Jamet]

It is possible to recognize whether a set of points in $\mathbb{Z}^d$ is contained in a stepped surface by considering a finite neighbour of each point.

Action on a plane

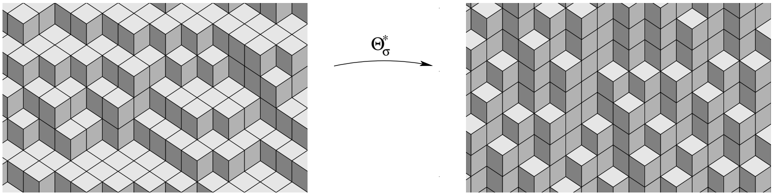
Theorem [Arnoux-Ito, Fernique]

Let σ be a unimodular substitution. Let $\vec{\alpha} \in \mathbb{R}_+^d$ be a nonzero vector. The generalized substitution Θ_σ^* maps without overlaps the stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ onto $\mathcal{P}_{{}^t M_\sigma \vec{\alpha},\rho}$.



Theorem [Arnoux-B.-Fernique-Jamet 2007]

Let σ be a unimodular subtitution. The generalized substitution Θ_σ^* acts without overlaps on stepped surfaces.

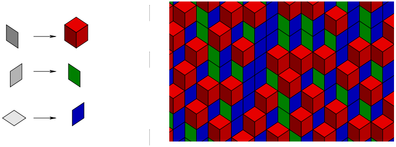


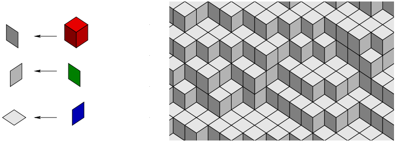
Tiling

Definition

Let σ be a unimodular substitution. A stepped surface is said to be σ-tilable if it is a union of translates of $\Theta_\sigma^*(\vec{0}, i^*)$.

Question: Can we desubstitute a σ-tilable stepped surface?





Desubstitution

We want to desubstitute a stepped surface according to Θ_σ^*.

Theorem

Let σ be an invertible substitution. Let $\mathcal{S}$ be a σ-tilable stepped surface. There exists a unique stepped surface $\mathcal{S}'$ such that

$$\Theta_\sigma^*(\mathcal{S}') = \mathcal{S}.$$

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Idea of the proof:

1. $\Theta_{\sigma\circ\mu}^* = \Theta_\mu^* \circ \Theta_\sigma^*$
2. $\Theta_{\sigma^{-1}}^*(\mathcal{S})$ is a stepped surface.
3. $\Theta_{\sigma^{-1}}^*(\mathcal{S})$ is thus an antecedent of $\mathcal{S}$ under the action of Θ_σ^*.

Desubstitution

Property

Let σ be a unimodular morphism of the free group. Let $\mathcal{S}$ be a σ-tilable stepped surface. Then $\Theta^*_{\sigma^{-1}}(\mathcal{S})$ is a stepped surface.

We use the following fact:

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Then, Θ^*_σ maps without overlaps the stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ onto $\mathcal{P}_{{}^t M_\sigma \vec{\alpha},\rho}$.

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Then, Θ^*_σ maps without overlaps the stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ onto $\mathcal{P}_{{}^t M_\sigma \vec{\alpha},\rho}$.

The stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ is σ-tilable iff

$$ {}^t M_{\sigma^{-1}} \vec{\alpha} \geq 0. $$

Brun's algorithm ($d = 2$)

We consider the following transformation acting on $[0,1]^2$

$$T(\alpha,\beta) = \begin{cases} (\{1/\alpha\}, \frac{\beta}{\alpha}) & \text{if } \alpha \geq \beta \\ (\frac{\alpha}{\beta}, \{1/\beta\}) & \text{otherwise.} \end{cases}$$

For all $n \in \mathbb{N}$, we set $(\alpha_n, \beta_n) = T^n(\alpha,\beta)$.
One has $(1, \alpha_n, \beta_n) \propto B_{a_n,\varepsilon_n}(1, \alpha_{n+1}, \beta_{n+1})$ with

$$(a_n, \varepsilon_n) = \begin{cases} (\lfloor 1/\alpha_n \rfloor, 1) & \text{if } \alpha_n \geq \beta_n \\ (\lfloor 1/\beta_n \rfloor, 2) & \text{otherwise,} \end{cases}$$

with

$$B_{a,1} = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{a,2} = \begin{pmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Continued fraction algorithm

One thus gets

$$\vec{\alpha} = \vec{\alpha}_0 \xrightarrow{B_0} \vec{\alpha}_1 \xrightarrow{B_1} \dots \xrightarrow{B_{n-1}} \vec{\alpha}_n \xrightarrow{B_n} \dots$$

where $B_n \in GL(d+1, \mathbb{N})$.

Convergents

$$\begin{aligned} (1, \vec{\alpha}) &\propto B_0 \times \dots \times B_n(1, \vec{\alpha}_{n+1}) \\ (q_n, \vec{p}_n) &\propto B_0 \times \dots \times B_n(1, \vec{0}). \end{aligned}$$

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- Unimodular algorithm
- Weak convergence (convergence of the type $|\alpha - p_n/q_n|$)

Arithmetics	Geometry
d-uple $\vec{\alpha} \in [0,1]^d$	stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$
$(1,\vec{\alpha}_n) = B_n(1,\vec{\alpha}_{n+1})$	$\mathcal{P}_{(1,\vec{\alpha}_n)} = \Theta^*_{\sigma_n}(\mathcal{P}_{(1,\vec{\alpha}_{n+1})})$ with ${}^tB_n$ incidence matrice of σ_n

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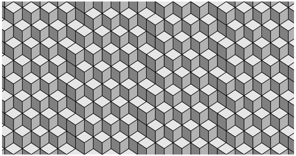
$(1, \alpha_0, \beta_0) = (1, \frac{11}{14}, \frac{19}{21})$	

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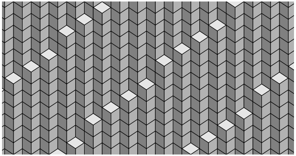
$(1, \frac{11}{14}, \frac{19}{21}) \propto B_{1,2}(1, \frac{33}{38}, \frac{2}{19})$	

<table>
<tr><th>Arithmetics</th><th>Geometry</th></tr>
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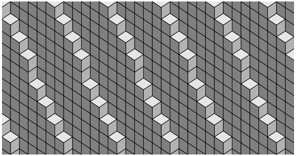
<tr><td>$(1, \frac{33}{38}, \frac{2}{19}) \propto B_{1,1}(1, \frac{5}{33}, \frac{4}{33})$</td><td></td></tr>
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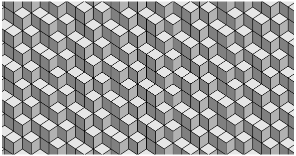
$(1, \frac{5}{33}, \frac{4}{33}) \propto B_{6,1}(1, \frac{3}{4}, \frac{4}{5})$	

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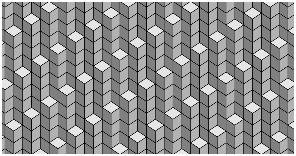
$(1, \frac{3}{4}, \frac{4}{5}) \propto B_{1,2}(1, \frac{3}{4}, \frac{1}{4})$	

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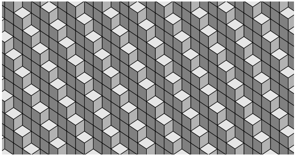
$(1, \frac{3}{4}, \frac{1}{4}) \propto B_{1,1}(1, \frac{1}{3}, \frac{1}{3})$	

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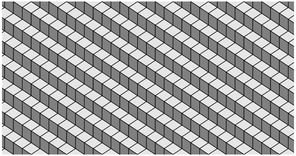
$(1, \frac{1}{3}, \frac{1}{3}) \propto B_{3,1}(1,0,1)$	

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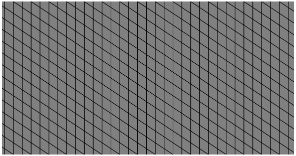
$(1,0,1) \propto B_{1,2}(1,0,0)$	

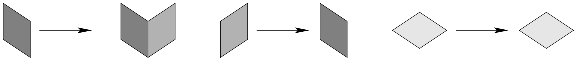
Choice of the substitution

Property

Let σ be a unimodular morphism of the free group. The stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ is σ-tilable iff

$$ {}^tM_{\sigma^{-1}}\vec{\alpha} \geq 0. $$

This provides a discrete version of Brun's algorithm.



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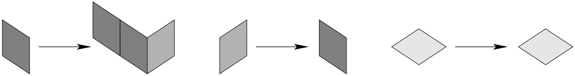

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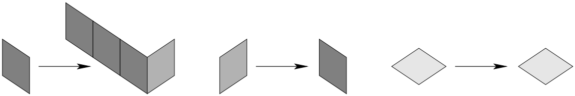

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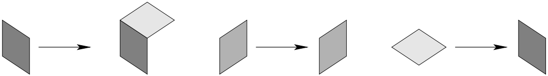

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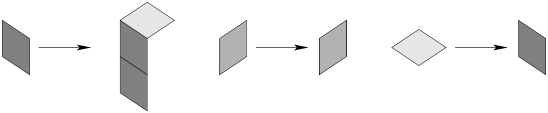

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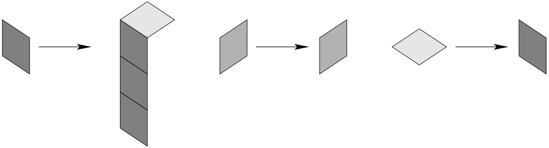

Conclusion

Theorem

If a stepped surface can be desubstituted infinitely many times according to Brun's algorithm, then it is a stepped plane with parameters given by the corresponding Brun's expansion.

Further work

- Higher codimensions (Penrose tilings)
- Finite case
- Multidimensional Ostrowski numeration based on Brun's algorithm
- Rauzy fractals in the S-adic case