Combinatorial and arithmetical properties of infinite words associated with quadratic non-simple Parry numbers

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- Terminology of arithmetics on β-integers $\mathbb{Z}_\beta$

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- Improvement of the upper bound on $L_\oplus(\beta)$, i.e., on fractional part arising when two β-integers are added

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β-expansion and β-integers

- Let $\beta > 1$ and $x \geq 0$, any series $x = \sum_{i=-\infty}^{k} x_i \beta^i, \;\; x_i \in \mathbb{N}_0$, is called a *$\beta$-representation* of x and denoted $x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots$.

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- $\mathbb{Z}_\beta$ is not closed under addition for $\beta \notin \mathbb{N}$!
- $\mathrm{Fin}(\beta)$ does not form a subring of $\mathbb{R}$ in general! (Finiteness property)

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- Formally written $L_\oplus(\beta) =$

$$\min\{L \in \mathbb{N}_0 \mid x, y \in \mathbb{Z}_\beta,\ x + y \in \mathrm{Fin}(\beta) \Rightarrow \mathrm{fp}_\beta(x+y) \leq L\}$$

if the set is not empty, otherwise $L_\oplus(\beta) := +\infty$.

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Rényi expansion of unity and Parry numbers

- Thurston: Let $d_\beta(1) = t_1 t_2 \ldots$, distances in $\mathbb{Z}_\beta$ form the set $\{\Delta_k \mid k \in \mathbb{N}_0\}$, where $\Delta_k := \sum_{i=1}^{\infty} \frac{t_{i+k}}{\beta^i}$.

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- $\{\Delta_k \mid k \in \mathbb{N}_0\}$ is finite $\Leftrightarrow$ $d_\beta(1)$ is eventually periodic.
- If $d_\beta(1)$ is eventually periodic, β is called a *Parry number*.
- If $d_\beta(1)$ is finite, β is called a *simple Parry number*.

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  - β is unit for $p - 1 = q$

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Improvement of the upper bound on $L_{\oplus}(\beta)$

- Parry condition: $d_\beta(1) = pq^\omega, \; p - 1 > q \geq 1$
  $x_k x_{k-1} \ldots x_0 \bullet x_{-1} \ldots$ is a β-expansion if and only if

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  $$x_i x_{i-1} \cdots \prec pq^\omega \quad \text{for all } i \leq k$$
- Any finite β-representation can be transformed to the β-expansion, which is also finite!

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  - or $x - y \notin \mathrm{Fin}(\beta)$.
- Subtraction of positive elements does not raise $L_\oplus(\beta)$.

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- There exists $\varepsilon \in \{0, \ldots, \lceil \frac{p}{q} \rceil\}$ such that $x + y \in \mathbb{Z}_\beta + \varepsilon \frac{p-q}{\beta}$.
- Theorem: Let $d_\beta(1) = pq^\omega$, $p - 1 > q \geq 1$, then

$$L_\oplus(\beta) \leq \lceil \frac{p}{q} \rceil.$$

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Infinite word u_β associated with β-integers

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- Fabre: Associate $\Delta_0 \to A$ and $\Delta_1 \to B$, you get a right-sided infinite word u_β, fixed point of the substitution

$$\varphi(A) = A^p B, \ \varphi(B) = A^q B$$

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Balance property

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  - Turek: The lowest possible c for u_β associated with quadratic simple Parry numbers.

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- u_β is $\lceil \frac{p}{q} \rceil$-balanced (using arithmetics)
- the number of A's in any prefix of u_β is greater or equal to the number of A's in any other factor of u_β of the same length
- u_β is $\lceil \frac{p-1}{q} \rceil$-balanced, which is the best possible upper bound (using combinatorial techniques)

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Deduction of the lower bound on $L_{\oplus}(\beta)$

- The precise balance property implies that there exists a prefix $\hat{w}$ and a factor w of u_β such that $|\hat{w}|_A - |w|_A = \lceil \frac{p-1}{q} \rceil$.

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- Let $x < y$ be the β-integers corresponding to w and z the end-point of $\hat{w}$, then $x + z = y + \lceil \frac{p-1}{q} \rceil (\Delta_0 - \Delta_1) = y + \lceil \frac{p-1}{q} \rceil \frac{p-q}{\beta}$.

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- It follows that $\mathrm{fp}_\beta(x+z) \geq \mathrm{fp}_\beta(\lceil \frac{p-1}{q} \rceil \frac{p-q}{\beta}) \geq \lfloor \frac{p-1}{q} \rfloor$.

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- $\lceil \frac{p}{q} \rceil - \lfloor \frac{p-1}{q} \rfloor = 1$, we conjecture that $L_\oplus(\beta) = \lfloor \frac{p-1}{q} \rfloor$.