Defects of fixed points of substitutions

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Journées de Numération

Graz, April 2007

Outline

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2. Beta-numeration
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   - Known results
   - Conjecture of Hof, Knill and Simon

Defects in words
Definitions and examples

[Droubay et al.] Every finite word w contains at most $|w| + 1$ palindromes.

Definition

The difference between $|w| + 1$ and the actual number of palindromes is called defect.

Example

$$0100 : \varepsilon$$
$$0120 :$$

Definition

- A finite word with zero defect is called full.
- An infinite word is said to be full if all its prefixes are full.

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Defects in words
Property *Ju*

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A finite word w satisfies the Property *Ju* if its longest palindromic suffix occurs exactly once in w.

Proposition (Droubay, Justin, Pirillo)

A finite word is full if and only if each its prefix satisfies Ju.

Known results

- Sturmian words are full
- Episturmian words are full

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Necessary condition

N.C.: If an infinite word is full then it contains an infinite number of palindromes.

Lemma (A., Frougny, Masáková, Pelantová)

If an infinite uniformly recurrent word contains an infinite number of palindromes then its language is closed under reversal.

- The language of an infinite word is the set of all its factors.
- A language is closed under reversal if with every word $w_1 \cdots w_k$ it contains also $w_k \cdots w_1$.

Remark. Berstel et al. gave an example showing that the converse of the lemma is not true.

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Beta-expansions
Beta-transformation

Beta-transformation $T_\beta : [0,1) \to [0,1)$ given by $T_\beta(x) := \beta x \pmod 1$.

Definition

Rényi expansion of 1 $\mathrm{d}_\beta(1) = (t_i)_i \geq 1$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$.

- If $\mathrm{d}_\beta(1)$ is eventually periodic, β is called Parry number
- If $\mathrm{d}_\beta(1)$ is finite, β is called simple Parry number

Remark. $\mathrm{d}_\beta(1)$ can be used to characterize expansions in β-numeration system.

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With each $\beta > 1$ one can associate an infinite word u_β, coding distances between β-integers in the associated β-numeration system.

u_β can be obtained as the unique fixed point $u_\beta = \lim_{n\to\infty} \varphi_\beta^n(0)$ of the β-substitution φ_β.

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Simple Parry case, $\mathrm{d}_\beta(1) = t_1 \cdots t_m$

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\begin{aligned}
\varphi_\beta(0) &= 0^{t_1}1 \\
\varphi_\beta(1) &= 0^{t_2}2 \\
&\vdots \\
\varphi_\beta(m-2) &= 0^{t_{m-1}}(m-1) \\
\varphi_\beta(m-1) &= 0^{t_m}
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Non-simple Parry case, $\mathrm{d}_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})$

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$$\vdots$$

$$\varphi_\beta(m+p-2) = 0^{t_{m+p-1}}(m+p-1)$$

$$\varphi_\beta(m+p-1) = 0^{t_{m+p}}(m)$$

Palindromes and defects in u_β

Necessary condition

The language of u_β is closed under reversal if and only if

- $\mathrm{d}_\beta(1) = t \cdots ts = t^k s$ for simple Parry number β,
- $\mathrm{d}_\beta(1) = ts^\omega$ for non-simple Parry number β.

Lemma (Simple Parry, $\mathrm{d}_\beta(1) = t^k s$)

- *A factor p of u_β is a palindrome iff $\varphi(p)0^t$ is a palindrome.*
- *For every palindrome p (not equal to 0^r, $r \leq t$), there exists a unique shorter palindrome q such that p occurs only as a central factor of $\varphi(q)0^t$.*

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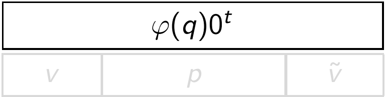

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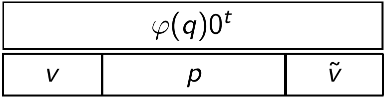

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Results

Theorem

u_β is full both in the simple and non-simple Parry case.

Proof. (non-simple case)

- v be the shortest prefix of u_β not satisfying Ju, *i.e.* its longest palindromic suffix occurs at least twice

$$u_\beta = v \cdots = upwp \cdots$$

- by Lemma, p occurs only as a central factor of $1\varphi(q)$, q palindrome, $|q| < |p|$

$$u_\beta = v \cdots = \varphi(\hat{u}q\hat{w}q) \cdots$$

- $\hat{u}q\hat{w}q$ is prefix containing twice its longest palindromic suffix q

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Palindromes and defects in general

Results

Similar techniques as for u_β works also for

- Period doubling word, fixed point of $\varphi(0) = 01$, $\varphi(1) = 00$
- Rote word, foxed point $\varphi(0) = 001$, $\varphi(1) = 111$

Not everything is full!

Thue-Morse word, fixed point of $\varphi(0) = 01$, $\varphi(1) = 10$

$$u_T = 011010011|0010110\cdots$$

Note that

- Thue-Morse word contains an infinite number of palindromes
- It has the "nice properties" similar to lemmas, e.g. $p \in \mathcal{P}al(u_{TM}) \Leftrightarrow \varphi^2(p) \in \mathcal{P}al(u_{TM})$

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Theorem (Brlek et al.)

Let $w = uv$, $|u| > |v|$, u,v palindromes. Then the defect of w^ω is bounded by the defect of its prefix of length $|uv| + \left\lfloor \frac{|u|-|v|}{3} \right\rfloor$.

Palindromes and defects

Infinite number of palindromes

Conjecture (Hof, Knill, Simon)

If a uniformly recurrent word u, fixed point of a morphism, contains infinitely many palindromes then there exist a morphism φ, a palindrome p and palindromes q_a such that u is a fixed point of φ and for every letter a one has

$$\varphi(a) = pq_a\,.$$

Remark.

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