Coding of irrational rotation: a different view

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Joint work with **M. Shirasaka**.

Let $\mathcal{A} = \{0, 1, \ldots, m-1\}$ be a finite set of letters and $\mathcal{A}^*$ be the monoid over $\mathcal{A}$ generated by concatenation, having the identity element λ, the empty word. The set of right infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$.

Let $\mathcal{C}$ a non empty finite set. A **morphism** σ is a monoid homomorphism from $\mathcal{A}^*$ to $\mathcal{C}^*$ that $\sigma(a) \neq \lambda$ for each $a \in \mathcal{A}$. Then σ naturally extends to a map from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{C}^{\mathbb{N}}$. A morphism σ is called **letter to letter**, if $\sigma(a) \in \mathcal{C}$ for each $a \in \mathcal{A}$. A **substitution** is a morphism from $\mathcal{A}^*$ to itself.

Identify $[0,1)$ with the torus $\mathbb{R}/\mathbb{Z}$. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$. Divide $[0,1)$ into

$$I_0 \cup I_1 = [0, 1-\xi) \cup [1-\xi, 1)$$

A **Sturmian word** is a coding of an irrational rotation $x \mapsto x+\xi$ with an initial value μ given by

$$J(\mu)J(\mu+\xi)J(\mu+2\xi)\dots$$

where

$$J(x) = \begin{cases} 0 & x \in I_0 \\ 1 & x \in I_1 \end{cases}.$$

Any Sturmian word has **desubstitution** structure, i.e., we can decompose them into one of the four larger blocks $\{01,0\},\{01,1\},\{10,0\}$ or $\{10,1\}$ which also gives a new Sturmian word.

$$\begin{aligned} 1001010010010100101 0\ldots &= \overline{10}\,0\,\overline{10}\,\overline{10}\,0\,\overline{10}\,0\,\overline{10}\,\overline{10}\,0\,\overline{10}\,\overline{10}\cdots \\ &= 010010100100101001010\ldots \end{aligned}$$

This property is extensively used to recode Sturmian words (c.f. Chapter 6 by Arnoux in [4] on Sturmian sequence). We wish to generalize this property.

An infinite word $z \in \mathcal{A}^{\mathbb{N}}$ is **recursively k-renewable** if there is a sequence of substitutions $\{\phi_i\}$ on $\mathcal{A} = \{0, 1, \dots k-1\}$ which are not letter to letter such that

$$z = z_0 \xleftarrow{\phi_1} z_1 \xleftarrow{\phi_2} z_2 \xleftarrow{\phi_3} \dots$$

with $z_i \in \mathcal{A}^{\mathbb{N}}$. Thus a recursively k-renewable word belongs to the inverse limit $\varprojlim_{\phi_i} \mathcal{A}^{\mathbb{N}}$. Sturmian words are recursively 2-renewable.

Corrigendum (due to J. Cassaigne). To exclude trivial cases, we have to assume that each letter a and $m \in \mathbb{N}$ there is an n that $|\phi_{m+1}\phi_{m+2} \dots \phi_{m+n}(a)| > 1$.

Consider an arbitrary decomposition:

$$0 = \omega_0 < \omega_1 < \cdots < \omega_{k-1} < \omega_k = 1.$$

A **general rotation word** is a coding of the irrational rotation $x \mapsto x + \xi$ with the initial value μ given by

$$J(\mu)J(\mu + \xi)J(\mu + 2\xi)\ldots$$

with $J(x) = i$ when $x \in [\omega_i, \omega_{i+1})$.

Theorem 1.
The general rotation word with respect to a k-decomposition

$$0 = \omega_0 < \omega_1 < \cdots < \omega_{k-1} < \omega_k = 1.$$

is recursively $(k+1)$-renewable.

Example. A general rotation word of an angle $\xi = 2^{-2/3}$ and an initial value $\mu = 0$ with respect to a decomposition

$$0 < 1/3 < 1$$

is recursively 3-renewable:

$$
\begin{aligned}
& 01011011010110110111101101101011011010110110\ldots \\
= \; & \overline{01}\,\overline{01}\,1\,\overline{01}\,1\,\overline{01}\,\overline{01}\,1\,\overline{01}\,1\,\overline{01}\,\overline{11}\,1\,\overline{01}\,1\,\overline{01}\,1\,\overline{01}\,\overline{01}\,1\,\overline{01}\,1\,\overline{01}\,\overline{01}\,1\,\overline{01}\,1\,\overline{01}\ldots \\
= \; & 00202002020120202002020020201202020020200202\ldots \\
= \; & \overline{002}\,02\,\overline{002}\,02\,\overline{012}\,02\,02\,\overline{002}\,02\,\overline{002}\,02\,\overline{012}\,02\,02\,\overline{002}\,02\,\overline{002}\,02\,\overline{012}\ldots \\
= \; & 02021220202122020212202021220212202021220202\ldots
\end{aligned}
$$

Ideas of the proof of Theorem 1.

- Take successive induced systems $[0, \xi_n)$ where ξ_{n+1} is determined by the first return to $[0, \xi_{n-1}) \simeq \mathbb{R}/\xi_{n-1}\mathbb{R}$ of the rotation $x \mapsto x + \xi_n$. These are given by the **negative continued fraction** (NCF).

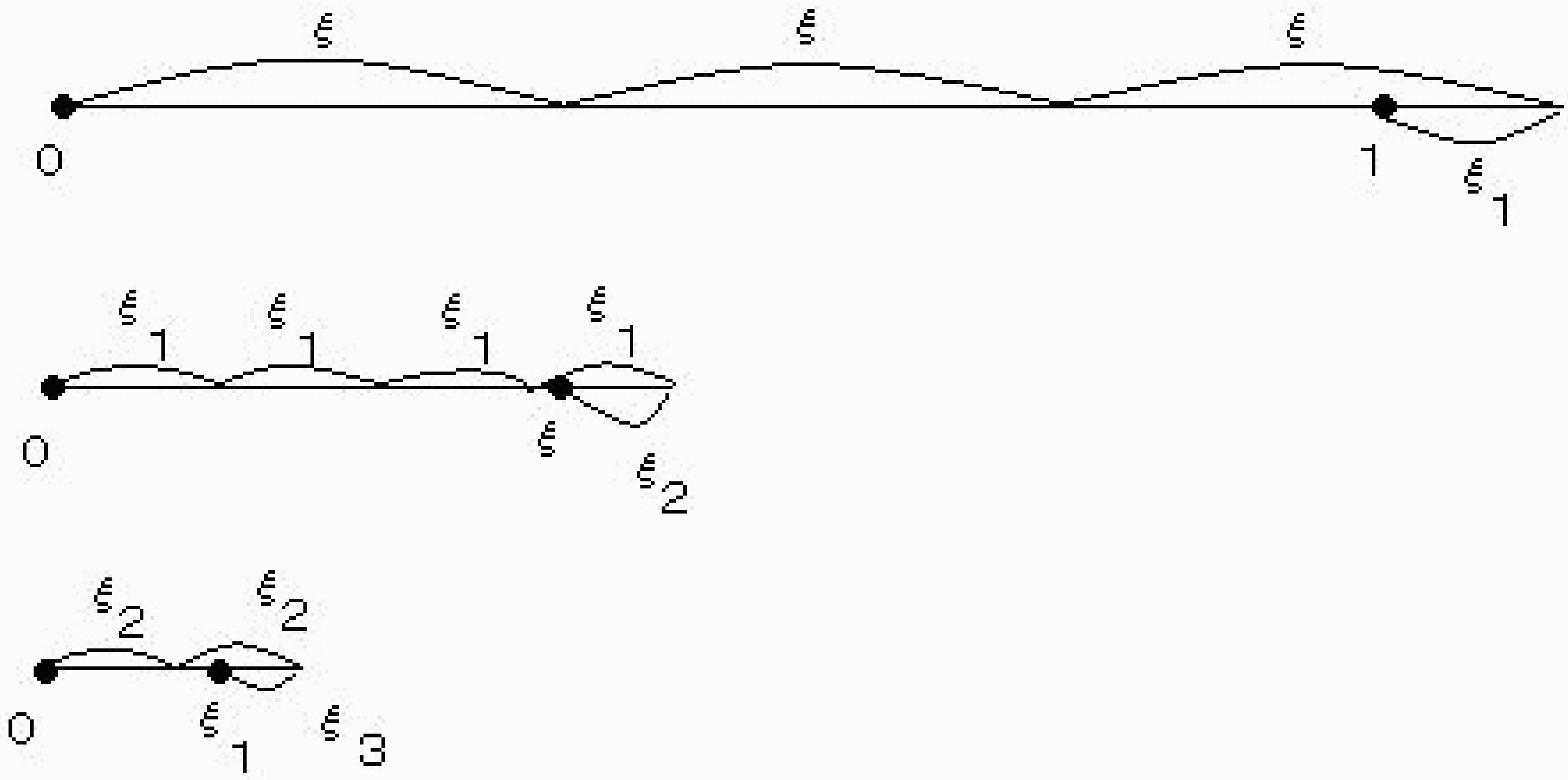
- **Dual Ostrowski numeration system** of NCF, that is, the greedy expansion with respect to base $\{\xi_n\}$, appears here.

- The 1-st induced system can have one more discontinuity.

- The 2-nd and later induced systems do not increase the number of discontinuities.

NCF

$$\xi = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n - \cfrac{\xi_{n+1}}{\xi_n}}}}}$$


ξ
ξ
ξ
0
1
ξ1
ξ1
ξ1
ξ1
ξ1
0
ξ
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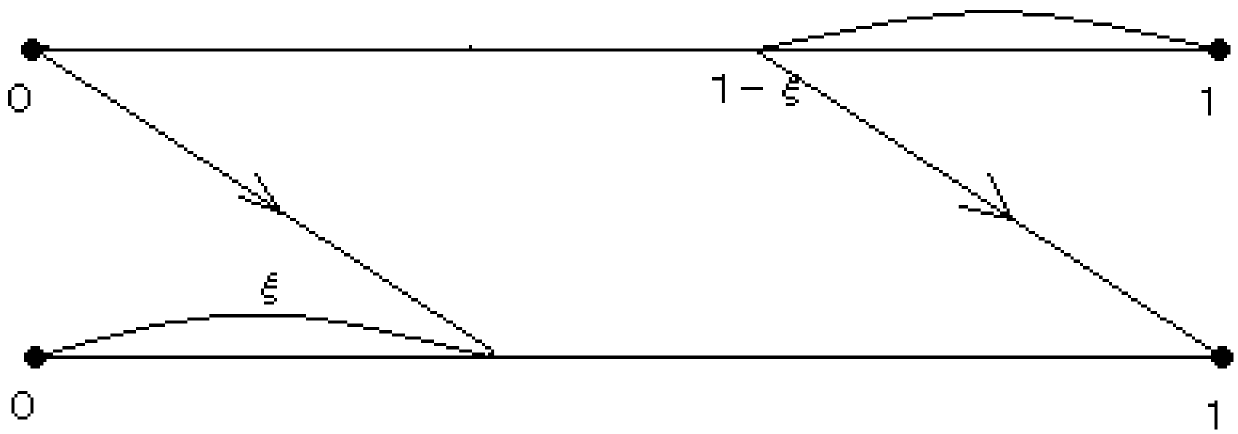
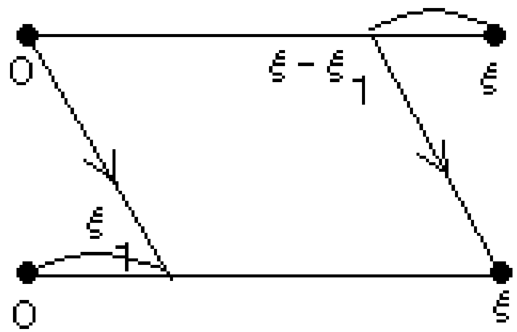
ξ2
ξ2
0
ξ1
ξ3


0
1 − ξ
1
ξ
0
1


0
ξ - ξ
1
ξ
ξ
1
0
ξ

A **stationary** recursive renewable words is also of interest, for e.g., the simplest case is

$$z \xleftarrow{\phi} z \xleftarrow{\phi} z \xleftarrow{\phi} \dots$$

using a single substitution ϕ. Among general rotation words, this concept corresponds to **substitution invariant** Sturmian words. The most famous example is the Sturmian word with $\xi = \mu = (3 - \sqrt{5})/2$

$$z = 010010100100101001010\dots$$

which is the fixed point of the Fibonacci substitution ϕ.(c.f. S.Yasutomi [6])

A substitution ϕ on $\mathcal{A}$ is **primitive**, if there is n such that $\phi^n(a)$ contains all letters of $\mathcal{A}$ for all $a \in \mathcal{A}$. A word $x \in \mathcal{A}^{\mathbb{N}}$ is **primitive substitutive** if it is an image of a morphism of a fixed point of a primitive substitution.

$$\text{Stationary} \Leftrightarrow \text{Eventually periodic } \{\phi_i\}$$
$$\Leftrightarrow \text{Primitive substitutive}$$

We can characterize primitive substitutive rotation words:

Theorem 2. The general rotation word of an angle ξ, an initial value μ with respect to a k-decomposition

$$0 = \omega_0 < \omega_1 < \cdots < \omega_{k-1} < \omega_k = 1.$$

is primitive substitutive if and only if ξ is quadratic irrational, $\mu \in \mathbb{Q}(\xi)$ and $\omega_i \in \mathbb{Q}(\xi)$.

This is a generalization of Adamczewski [1] and Berthé-Holton-Zamboni [2].

Ideas of the proof of Theorem 2.

- ‘If’ part is shown by a Pisot unit property of continued fraction.

- Rauzy induction is **not** used. (At least, apparently).

- To show the ‘only if’ part, we use the finiteness of derived words generated by **return words** due to Durand [3] and Holton-Zamboni [5].

Return words and derived words

Let z be a uniformly recurrent word and fix a prefix w. Consider the first return of w.

$$
\begin{aligned}
z &= 01011011010110110111101101101011011010110110\ldots \\
&= \mathbf{010}11011\mathbf{010}1101101111011011\mathbf{010}11011\mathbf{010}110110\ldots \\
&= \overline{01011011}\ \overline{0101101101111011011}\ \overline{01011011}\ \overline{0101101101111011011} \\
&= 0101\ldots(\text{Derived word})
\end{aligned}
$$

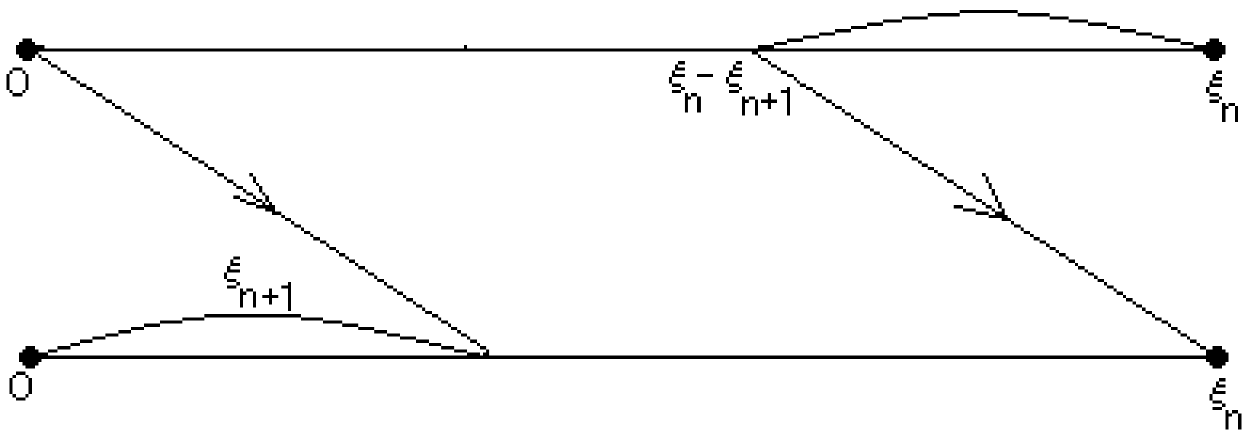
Durand [3] and Holton-Zamboni [5] proved that z is primitive substitutive if and only if the set of derived words is finite.

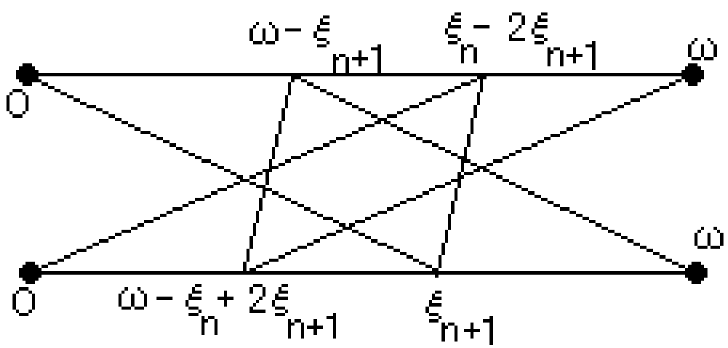
Ideas of the proof of Theorem 2. (Continued)

- We prove that return words with respect to a long prefix gives a coding of a certain three interval exchange.

- To have unique ergodicity of the three interval exchange, we need the precise behavior of induced discontinuities.

- For this purpose, we essentially use **Ostrowski numeration system** (original and its dual) with respect to the NCF.


0
ξn − ξn+1
ξn
ξn+1
0
ξn


ω − ξn+1
ξn − 2ξn+1
ω
0
ω
0
ω − ξn + 2ξn+1
ξn+1

Questions

- Can you characterize general rotation words among renewable words?
- Is the natural coding of IET recursively renewable ? (might be stupid)
- How to generalize Theorem 2 for other codings (IET or else) ?

References

[1] B. Adamczewski, *Codages de rotations et phénomènes d'autosimilarité.*, J. Théor. Nombres Bordeaux **14** (2002), no. 2, 351–386.

[2] V. Berthé, C. Holton, and L.Q. Zamboni, *Initial powers of sturmian sequences*, Acta Arith **122** (2006), no. 4, 315–347.

[3] F. Durand, *A characterization of substitutive sequences using return words*, Discrete Math. **179** (1998), no. 1-3, 89–101.

[4] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002, Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[5] C. Holton and L.Q. Zamboni, *Descendants of primitive substitutions*, Theory Comput. Syst. **32** (1999), no. 2, 133–157.

[6] S. Yasutomi, *On sturmian sequences which are invariant under some substitutions*, Number theory and its applications (Kyoto, 1997), Kluwer Acad. Publ., Dordrecht, 1999, pp. 347–373.